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A second-order, one strip shock, integral relations study of the direct non-equilibrium hypersonic blunt body problem

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A SECOND-ORDER, ONE STRIP SHOCK, INTEGRAL RELATIONS STUDY
OF THE DIRECT NON-EQUILIBRIUM HYPERSONIC BLUNT BODY PROBLEM

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LIST OF SYMBOLS

A	species density
c_i	species concentration
D	heat of dissociation
e	internal energy
e_i^o	heat of formation
h	enthalpy
H_t	stagnation enthalpy
j	dimension index (see Equation 4)
k	metric coefficients for arc length in curvilinear system
$K_d^{(i)}$	dissociation rate coefficient
$K_r^{(i)}$	recombination rate coefficient
K_E	equilibrium constant
L	characteristic body length
p	pressure
q	velocity
q_{∞_M}	maximum free stream adiabatic velocity
r	distance measured perpendicular to the axis of symmetry
R	gas constant
R_b	body radius of curvature
s	general curvilinear co-ordinate
T	temperature
U_∞	free stream velocity
u	velocity component along the body
v	velocity component normal to the body

x	distance measured along the body surface from the axis of symmetry
y	distance measured normal to the body surface from the surface
α	degree of dissociation
β	$K_r^{(2)} / K_r^{(1)}$
γ	specific heat ratio
Γ	see Equation 21
δ	shock layer thickness
ϵ	shock wave density ratio
θ	body surface angle
ϑ	body polar angle
λ	nonequilibrium parameter
μ	molecular weight
υ_i	chemical reaction coefficients
ρ	density
σ	shock wave angle
ϕ	body meridian angle
ψ	see Figure 3
ω_i	chemical source function
◺	see Equation 73
$\square$	see Equation 73
$\bigtriangledown$	see Equation 73
⦶	see Equation 73

Subscripts

A	refers to atomic species
b	on the body surface

r	rotational quantity
s	at the shock wave
t	translational quantity
v	vibrational quantity
o	stagnation point

INTRODUCTION

The blunt nosed re-entry vehicles used in present day ballistic weapon systems and space capsules return to the earth's atmosphere at extremely high velocities. A large portion of the kinetic energy associated with these high velocities is converted to thermal energy by the strong bow shock preceeding the vehicle. The resulting high temperature of the shock layer may cause excitation of the normally inert degrees of freedom of the gas including vibration, dissociation and ionization. The excitation of the internal degrees of freedom gives rise to real gas effects because the vibration, dissociation and ionization are not in equilibrium with the local temperature. The equilibration of these degrees of freedom generally takes a large number of molecular collisions which introduces a finite relaxation time. This relaxation time must be compared with some characteristic flow time to determine the magnitude of the departure from thermodynamic equilibrium. If this departure is sufficiently large, rate equations which describe the nonequilibrium effects must be included in the mathematical description of the gaseous media through which the body is flying. This paper deals with a technique for solving the hypersonic blunt body problem including nonequilibrium effects due to molecular dissociation.

The hypersonic blunt body problem is usually posed as the inverse or the direct problem. In the former, the shape of the bow shock is given and one is required to construct the generating body and the corresponding flow field. The direct problem specifies the body shape and the shock wave shape and flow field are to be determined. Several techniques have been used to obtain solutions to the direct and inverse problems. These are summarized in reference 13 and only a short discussion of the more important methods will be given here.

Specifying the shock shape in the inverse problem equivalently determines the variation of the flow properties and their normal derivatives along the shock surface. The Rankine-Hugoniot equations are used to determine the flow properties and the equations of motion specified to the shock provide the required derivatives. These known quantities are then used to initiate a step wise integration from the shock to the initially unknown body. Garabedian and Lieberstein (9) have used this approach for a perfect gas while Lick (17) has extended this to a bimolecular dissociating gas. Hall, Eschenroeder and Marrone (11) have used the inverse method with a complicated gas model including dissociation, vibration and ionization with the associated coupling effects to predict body shape and flow field characteristics corresponding to a given shock. The practical value of the inverse method must be questioned because one does not know the exact shock shape produced by a given body. This means that a large number of solutions for various shock shapes must be obtained for each operating altitude encountered. Even when a large number of solutions have been obtained, the solution for flow field and body shape are not generally the exactly desired results.

A much more practical method would be the direct solution as outlined previously. Many techniques, both exact and approximate have been used on the direct hypersonic blunt body problem. The only direct method that is considered exact is the stream tube continuity method in which the stream tubes in the shock layer are traced out by satisfying the conservation equations. The convergence properties of this approach have been questioned and as a result more effort is being expended on different techniques. The best known approximate methods are the Newtonian, constant density and thin shock theory solutions. The Newtonian theory assumes that the shock layer is very thin and has the same slope as the body. The normal component of momentum is then assumed to be lost inelastically and is transmitted to the body through the shock layer. The constant density and thin shock layer theories are of less importance but both use assumptions implied by their names. Unfortunately these methods give only order of magnitude results over the entire spectrum of altitudes and velocities of interest. The method of integral relations as introduced by Dorodnitsyn (6) and applied by Belotserkovskii (2) provides results that are as accurate as desired for any velocity and altitude. The only limit to the accuracy of this method is the effort in the formulation of higher-order approximations and digital computer time.

The technique used to solve the direct hypersonic blunt body problem in this paper is the method of integral relations. The first-order approximation for a dissociating gas using this technique has been formulated independently by both the author and Shih et al. (21) at the supersonic research laboratory at MIT. A second-order approximation using a one strip shock with a reacting gas is formulated in the following chapters. The formulation is made for a general axisymmetric or two-dimensional body with a general dissociating gas. Specification is then made to a sphere and a Lighthill gas (18) to demonstrate the technique.

THE GENERAL INTEGRAL RELATIONS METHOD FOR A REACTING GAS

General Remarks

The method of integral relations is based on a paper by Dorodnitsyn (6) in which he proposed a general technique for solving non-linear fluid mechanics problems. This technique has been applied to the hypersonic blunt body problem using a perfect gas and considerable effort is being expended to modify the approach for a real gas.

The equations of fluid dynamics are usually in a form which is not suited to the integral technique. Dorodnitsyn's method requires that this system of partial differential equations describing the fluid behavior be cast into a divergence form. This form is obtained by combining the equations of motion in a particular manner. Consider the following divergence form:

$$\frac{\partial P_i}{\partial x}(x, y; u_1, \ldots, u_n) + \frac{\partial G_i}{\partial y}(x,y; u_1, \ldots, u_n) = L_i(x,y; u_1, \ldots, u_n) \qquad i=1, \ldots, n \qquad (1)$$

where P_i, G_i and L_i are known functions of their arguments and u_i are unknown functions of the independent variables x,y. For simplicity, consider the domain to be rectangular in shape and let it be determined by $0 \leqslant x \leqslant$ constant and $0 \leqslant y \leqslant 1$. This may represent the shock layer of a blunt re-entry body after undergoing a suitable transformation of co-ordinates. This domain is divided into N strips by drawing lines $y_k = 1 - \frac{k-1}{N}$ (Figure 1) and the divergence Equation 1 is integrated from y=0 to $y=y_k$, the boundary of each strip.

$$(G_i)_k + \frac{d}{dx} \int_0^{y_k} P_i \, dy = \int_0^{y_k} L_i \, dy + (G_i)_{y=o} \qquad (2)$$

Each strip in the domain provides one integral relation. If there are n equations in the system, we have nN total relations of the form of Equation 2.

It is assumed that the integrands L_i and P_i may be expressed as polynomials of the form:

$$L_i = \sum_{j=0}^{N} a_{ij}(x)\, y^j \quad . \tag{3}$$

The a_{ij}'s depend linearly on the values of the L_i functions on the strip boundaries as implied in Equation 3. The use of the polynomial expansion permits the evaluation of the remaining integrals in Equation 2. This integration provides a system of nN ordinary differential equations in the dependent variables on the strip boundaries. These ordinary differential equations are then integrated from x=0 to x= constant to obtain solutions for u_i.

Specific Formulation for the Blunt Body Problem

The equations of motion of a compressible fluid flowing about an arbitrary two-dimensional or axisymmetric body are derived in the following pages. The analysis is simplified by using the following assumptions:

1) The gas is non-viscous and non-heat conducting.

2) Diffusion is neglected.

3) The gas is a mixture of thermally perfect gases.

4) The translational, rotational and vibrational degrees of freedom are those pertaining to thermal equilibrium.

5) Ionization and radiation are neglected.

Gibson (10) has shown that the major effect of diffusion is to make the assumption of frozen flow through the shock with respect to dissociation invalid and also to smooth or decrease the concentration gradients immediately behind the shock. This post shock concentration effect is most pronounced on electron and nitric oxide densities. Since this paper is concerned with a simple diatomic gas in which the electronic degrees of freedom are considered frozen, the diffusion effect is neglected.

Assumption 4 may be questioned in the light of recent developments (4). The inclusion of vibrational relaxation and the associated coupling between vibration and dissociation would only complicate the problem since the main purpose of this paper is to demonstrate the feasibility of applying the second order, one-strip shock integral method to a dissociating gas. Since the gas model is strictly dissociating and the electronic degrees of freedom are neglected, there could of course be no radiation.

The body under consideration in this paper may be either two dimensional or axisymmetric so long as the body surface is continuous and has a continuous first derivative. The coordinate system for such a body is shown in Figure 2. The origin is taken at the stagnation point while the x coordinate is distance along the body and the y axis is everywhere normal to the body. Arc length in this system is given by:

$$\begin{aligned} k_1\, ds_1 &= [1 + \frac{y}{R_b}]\, dx \\ k_2\, ds_2 &= dy \\ k_3\, ds_3 &= r^j\, d\phi = [r_b + y\cos\theta]^j\, d\phi \end{aligned} \tag{4}$$

where k_1, k_2 and k_3 are the necessary metric coefficients for arc length due to the curvilinear nature of the coordinate system. These coefficients are on the order of unity for most practical bodies. The radius of curvature of the body is denoted by R_b, r is the distance measured normal to the axis of symmetry and ϕ is a unit distance normal to the x,y plane in two-dimensional (j = 0) or the meridian angle in axisymmetric flow (j = 1).

The continuity equation in this system is:

$$\frac{\partial(\rho u k_3)}{\partial x} + \frac{\partial(\rho v k_1 k_3)}{\partial y} = 0 \tag{5}$$

where ρ is the density, u is the velocity along the body and v is the velocity normal to the body.

The two components of Euler's equation are:

$$\frac{u}{k_1}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{k_1}\frac{\partial k_1}{\partial y} = -\frac{1}{\rho k_1}\frac{\partial p}{\partial x} \tag{6}$$

$$\frac{u}{k_1}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{u^2}{k_1}\frac{\partial k_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{7}$$

where p is the fluid pressure.

The energy equation is

$$\frac{u}{k_1}\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - \frac{1}{\rho}\left[\frac{u}{k_1}\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right] = 0 \quad . \tag{8}$$

Since the total energy remains constant along a streamline in steady adiabatic flow (16), this equation may also be written:

$$h + \frac{q^2}{2} = H_T = \text{constant} \tag{9}$$

where the enthalpy is defined as:

$$h = e + p/\rho \tag{10}$$

and the internal energy is given by:

$$e = \sum_i g_i\,(e_i - e_i^o) \quad . \tag{11}$$

Here g_i is the mass of the ith species per unit mass of gas mixture, e_i is specific internal energy of the ith species and e_i^o is the heat of formation of the ith species per unit mass. The summation is over all components of the gas. The internal energy for the ith species may be written:

$$e_i = e_{i_t} + e_{i_r} + e_{i_v} \tag{12}$$

where the translational, rotational and vibrational energies are given by:

$$\begin{aligned} e_{i_t} &= 3/2\, R_i T \\ e_{i_r} &= R_i T \\ e_{i_v} &= \frac{h\upsilon}{KT}\left[\frac{1}{\exp\frac{h\upsilon}{KT} - 1}\right] R_i T \end{aligned} \tag{13}$$

where R_i is the gas constant for the ith species, T is the temperature, h is Planck's constant, k is Boltzmann's constant and υ is the vibrational frequency of the diatomic molecules. The energies given in Equation 13 have been written explicitly for a diatomic gas and are derived from statistical mechanics (8).

Dalton's law is assumed to hold and the equation of state for a mixture of perfect gases may be written as:

$$p = \sum_i \frac{g_i}{\mu_i} \rho RT \tag{14}$$

where μ_i is the molecular weight of the ith species, p is pressure, R is the universal gas constant and ρ is the density of the mixture.

If the gas is dissociating, the reaction equations describing this phenomenon and the rate equation for each species must be written. Then the equation describing a one step chemical reaction can be written (20):

$$\sum_{j=1}^{m} \upsilon_j' A_j \underset{K_r}{\overset{K_d}{\rightleftharpoons}} \sum_{j=1}^{m} \upsilon_j'' A_j \tag{15}$$

where the υ_j's are stoichiometric coefficients of the reactants and products and the summation is over all species entering the reaction. The K's are the dissociation and recombination rate coefficients.

The result of the chemical reaction 15 is to produce a net change in concentration of each constituent of the gas. The rate or species continuity equation governing this change in concentration is:

$$\frac{\mathcal{D} c_i}{\mathcal{D} t} = \frac{\omega_i}{\rho} \quad , \tag{16}$$

where c_i is the mass of species i per unit mass of mixture and ω_i is the net production of the ith species. The $\mathcal{D}/\mathcal{D}t$ denotes the usual substantial derivative. By means of Equation 15, ω_i may be written (20):

$$\omega_i = \mu_i(\upsilon_i'' - \upsilon_i')K_d \prod_{j=1}^{m} (A_j)^{\upsilon_j'} + \mu_i(\upsilon_i' - \upsilon_i'')K_r \prod_{j=1}^{m} (A_j)^{\upsilon_j''} \quad . \tag{17}$$

This equation must be summed over all chemical reactions which give rise to a net change in the given species. If there are four independent chemical reactions then the net production of each reaction must be accounted for in ω_i.

The equations of motion must be cast into the divergence form of Equation 1 by combining them in a suitable manner. The continuity Equation 5 is already in proper form for the integral method. Multiplying Equation 7 by $\rho k_1 k_3$ and Equation 5 by v and adding, the following divergence form is obtained:

$$\frac{\partial}{\partial y} [k_1 k_3 (p + \rho v^2)] + \frac{\partial}{\partial x} [k_3 \rho uv] = k_3 (p + \rho u^2) \frac{\partial k_1}{\partial y} + p k_1 \frac{\partial k_3}{\partial y} \quad . \tag{18}$$

A similar technique applied to Equation 6 yields:

$$\frac{\partial}{\partial x}[k_3(p+\rho u^2)] + \frac{\partial}{\partial y}[k_1 k_3 \rho uv] + k_3 \rho uv \frac{\partial k_1}{\partial y} - p\frac{\partial k_3}{\partial x} = 0 \qquad (19)$$

and the rate equation for the ith species:

$$\frac{\partial}{\partial x}[\rho u k_3 c_i] + \frac{\partial}{\partial y}[\rho v k_1 k_3 c_i] = \rho k_1 k_3 \omega_i \quad . \qquad (20)$$

The governing equations can now be non-dimensionalized using the following substitutions:

$$\bar{u} = \frac{u}{q_{\infty_M}} \qquad \bar{v} = \frac{v}{q_{\infty_M}} \qquad \bar{\rho} = \frac{\rho}{\rho_{\infty_t}} \qquad \bar{p} = \frac{p}{p_{\infty_t}}$$

$$\bar{x} = \frac{x}{L} \qquad \bar{y} = \frac{y}{L} \qquad \bar{r} = \frac{r}{L} \qquad \bar{R} = \frac{R}{L}$$

$$\bar{h} = \frac{h}{2H_t} \qquad \Gamma = \frac{\gamma_\infty - 1}{2\gamma_\infty} \qquad (21)$$

where $q_{\infty_M} = \sqrt{2H_t}$ and the subscript ∞_t refers to free stream stagnation values and L is a characteristic body length.

Equation 5 and Equations 18 to 20 in non-dimensional form are:

$$\frac{\partial}{\partial \bar{x}}[\bar{r}^j \bar{\rho}\bar{u}] + \frac{\partial}{\partial \bar{y}}[(1+\frac{\bar{y}}{\bar{R}_b})\bar{r}^j \bar{\rho}\bar{v}] = 0 \qquad (22)$$

$$\frac{\partial}{\partial \bar{y}}[(1+\frac{\bar{y}}{\bar{R}_b})\bar{r}^j(\Gamma\bar{p}+\bar{\rho}\bar{v}^2) + \frac{\partial}{\partial \bar{x}}[\bar{r}^j\bar{\rho}\bar{u}\bar{v}] - \frac{\bar{r}^j}{\bar{R}_b}[\Gamma\bar{p}+\bar{\rho}\bar{u}^2] - j(1+\frac{\bar{y}}{\bar{R}_b})\Gamma p\cos\theta = 0 \qquad (23)$$

$$\frac{\partial}{\partial \bar{x}}[\bar{r}^j(\Gamma\bar{p}+\bar{\rho}\bar{u}^2)] + \frac{\partial}{\partial \bar{y}}[(1+\frac{\bar{y}}{\bar{R}_b})\bar{r}^j\bar{\rho}\bar{u}\bar{v}] + \bar{r}^j\frac{\bar{\rho}\bar{u}\bar{v}}{\bar{R}_b} - j\frac{\gamma_\infty - 1}{2\gamma_\infty}\bar{p}[\frac{d\bar{r}_b}{d\bar{x}} - \bar{y}\sin\theta\frac{d\theta}{d\bar{x}}] = 0 \qquad (24)$$

$$\frac{\partial}{\partial \bar{x}}[\bar{\rho}\bar{u}\bar{r}^j c_i] + \frac{\partial}{\partial \bar{y}}[\bar{\rho}\bar{v}(1+\frac{\bar{y}}{\bar{R}_b})\bar{r}^j c_i] = \bar{\rho}(1+\frac{\bar{y}}{\bar{R}_b})\bar{r}^j\bar{\omega}_i \quad . \qquad (25)$$

The general integral method now requires that each of the above equations be integrated from y = 0 to the boundary of each strip. Belotserkovskii has used this technique with both a two and three strip shock layer (2). His analysis was based on a perfect gas but the trend indicated in the solutions should be applicable to the case of other gas models. The results he obtained indicate that the two or three strip shock layer is necessary at low Mach numbers; while at the velocities of interest here, the one, two and three strip shock layers give practically the same results. This paper is based on a one strip model so the divergence form of the equations of motion are now integrated from the body to the shock.

The partially integrated forms of Equations 22 to 25 become:

$$\frac{d}{d\bar{x}} \int_0^{\bar{\delta}} \bar{r}^j \bar{\rho}\bar{u} d\bar{y} - [\bar{r}_s^j \bar{\rho}_s \bar{u}_s] \frac{d\bar{\delta}}{d\bar{x}} + [1 + \frac{\bar{\delta}}{\bar{R}_b}][\bar{r}_s^j \bar{\rho}_s \bar{v}_s] = 0 \tag{26}$$

$$\frac{d}{d\bar{x}} \int_0^{\bar{\delta}} \bar{r}^j \bar{\rho}\bar{u}\bar{v} d\bar{y} + [1 + \frac{\bar{\delta}}{\bar{R}_b}][\bar{r}_s^j (\Gamma \bar{p}_s + \bar{\rho}_s \bar{v}_s^2)] - \bar{r}_b^j \Gamma \bar{p}_b - \bar{r}_s^j \bar{\rho}_s \bar{u}_s \bar{v}_s \frac{d\bar{\delta}}{d\bar{x}}$$

$$- \int_0^{\bar{\delta}} [\frac{\bar{r}^j}{\bar{R}_b} (\Gamma \bar{p} + \bar{\rho}\bar{u}^2) + j \Gamma (1 + \frac{\bar{y}}{\bar{R}_b}) \cos \theta \, \bar{p}] d\bar{y} = 0 \tag{27}$$

$$\frac{d}{d\bar{x}} \int_0^{\bar{\delta}} [\bar{r}^j (\Gamma \bar{p} + \bar{\rho}\bar{u}^2)] d\bar{y} - [\bar{r}_s^j (\Gamma \bar{p}_s + \bar{\rho}_s \bar{u}_s^2)] \frac{d\bar{\delta}}{d\bar{x}} + [1 + \frac{\bar{\delta}}{\bar{R}_b}][\bar{r}_s^j \bar{\rho}_s \bar{u}_s \bar{v}_s]$$

$$+ \int_0^{\bar{\delta}} \frac{\bar{r}^j \bar{\rho}\bar{u}\bar{v}}{\bar{R}_b} d\bar{y} - j\Gamma \int_0^{\bar{\delta}} \bar{p} [\frac{d\bar{r}_b}{d\bar{x}} - \bar{y} \sin \theta \frac{d\theta}{d\bar{x}}] d\bar{y} = 0 \tag{28}$$

$$\frac{d}{d\bar{x}} \int_{0}^{\bar{\delta}} \bar{\rho}\bar{u}\bar{r}^{j} c_i \, d\bar{y} = \int_{0}^{\bar{\delta}} \bar{\rho} \left(1 + \frac{\bar{y}}{\bar{R}_b}\right) \bar{r}^{j} \bar{\omega}_i \, d\bar{y} \quad . \tag{29}$$

The development presented thus far is usually referred to as the first-order approximation. For this first-order approximation a set of total differential equations, which must be satisfied along the body, is obtained in terms of the conditions prevailing at the shock. These shock conditions enter the describing differential equations with the exception of the x-momentum and rate equations. The shock conditions do not appear in these equations because their exact forms are known along the body surface and the integration indicated in Equations 28 and 29 is not required. The proper forms to use on the body are:

$$\bar{u}_b \frac{d\bar{u}_b}{d\bar{x}} = - \Gamma \frac{1}{\bar{\rho}_b} \frac{d\bar{p}_b}{d\bar{x}} \tag{30}$$

and

$$\bar{u}_b \bar{\rho}_\infty \frac{dc_{i_b}}{d\bar{x}} = \bar{\omega}_{i_b} \quad . \tag{31}$$

The first-order approximation derives its name from the fact that a one strip shock layer is used. The order of the one strip shock approximation can be increased only be retaining more terms in the interpolation polynomials. In so far as the author knows, up to this time only a linear representation of the integrand functions has been used with the method of integral relations regardless of the number of divisions in the shock layer. When the first two terms are used to represent the function across the shock layer, the coefficients depend linearly on the values of the function on the strip boundaries. For a one strip, linear case, the first integrand of Equation 26 would read:

$$\bar{r}^j\bar{\rho}\bar{u} = \bar{r}_b^j\ \bar{\rho}_b\bar{u}_b + [\bar{r}_s^j\ \bar{\rho}_s\bar{u}_s - \bar{r}_b^j\ \bar{\rho}_b\ \bar{u}_b]\ \frac{\bar{y}}{\bar{\delta}} \quad . \tag{32}$$

A second-order polynomial is used in the present study to approximate the integrand functions across the shock layer. For the second-order approach the same integrand reads:

$$\bar{r}^j\bar{\rho}\bar{u} = \bar{r}_b^j\bar{\rho}_b\bar{u}_b + [2(\bar{r}_s^j\ \bar{\rho}_s\bar{u}_s - \bar{r}_b^j\ \bar{\rho}_b\bar{u}_b) - \bar{\delta}\ (\frac{\partial(\bar{r}^j\bar{\rho}\bar{u})}{\partial\bar{y}})_s]\ \frac{\bar{y}}{\bar{\delta}}$$

$$+ [\bar{\delta}\ (\frac{\partial\bar{r}^j\bar{\rho}\bar{u}}{\partial\bar{y}})_s - (\bar{r}_s^j\ \bar{\rho}_s\bar{u}_s - \bar{r}_b^j\ \bar{\rho}_b\bar{u}_b)]\ (\frac{\bar{y}}{\bar{\delta}})^2 \quad . \tag{33}$$

The coefficients of this expansion depend on both the values of the functions and their derivatives at the shock. These first derivatives must be obtained from the equations of motion.

The expanded form of the continuity Equation 5 applied at the shock wave reads:

$$[1+\frac{\bar{\delta}}{\bar{R}_b}](\frac{\partial\bar{v}}{\partial\bar{y}})_s\ \bar{\rho}_s - \bar{\rho}_s\ \frac{d\bar{\delta}}{d\bar{x}}\ (\frac{\partial\bar{u}}{\partial\bar{y}})_s + [(1+\frac{\bar{\delta}}{\bar{R}_b})\bar{v}_s - \bar{u}_s\ \frac{d\bar{\delta}}{d\bar{x}}][\frac{\partial\bar{\rho}}{\partial\bar{y}}]_s = \Delta_s \tag{34}$$

where

$$\Delta_s = -[\frac{d\bar{\rho}\bar{u}}{d\bar{x}}]_s - \frac{\bar{\rho}_s\bar{v}_s}{\bar{r}_s^j}\ [j(1+\frac{\bar{\delta}}{\bar{R}_b})\cos\ \theta + \frac{\bar{r}^j}{\bar{R}_b}] - j\ \frac{\bar{\rho}_s\bar{u}_s}{\bar{r}_s^j}\ [\frac{d\bar{r}_b}{d\bar{x}} - \bar{\delta}\ \sin\ \theta\ \frac{d\theta}{d\bar{x}}] \tag{35}$$

Equation 6 becomes

$$\bar{\rho}_s\ [(1+\frac{\bar{\delta}}{\bar{R}_b})\bar{v}_s - \bar{u}_s\ \frac{d\bar{\delta}}{d\bar{x}}][\frac{\partial\bar{u}}{\partial\bar{y}}]_s - \frac{d\bar{\delta}}{d\bar{x}}\ \Gamma\ [\frac{\partial\bar{p}}{\partial\bar{y}}]_s = -\frac{\bar{\rho}_s\bar{u}_s\bar{v}_s}{\bar{R}_b} - \bar{\rho}_s\bar{u}_s[\frac{d\bar{u}}{d\bar{x}}]_s - \Gamma[\frac{d\bar{p}}{d\bar{x}}]_s \tag{36}$$

Equation 7 reads

$$\bar{\rho}_s\ [(1+\frac{\bar{\delta}}{\bar{R}_b})\ \bar{v}_s - \bar{u}_s\ \frac{d\bar{\delta}}{d\bar{x}}][\frac{d\bar{v}}{d\bar{y}}]_s + \Gamma\ [1+\frac{\bar{\delta}}{\bar{R}_b}][\frac{\partial\bar{p}}{\partial\bar{y}}]_s = \bar{\rho}_s\bar{u}_s[\frac{\bar{u}_s}{\bar{R}_b} - (\frac{d\bar{v}}{d\bar{x}})_s] \tag{37}$$

The energy Equation 8 may be written:

$$\frac{\bar{u}_s}{1+\frac{\bar{\delta}}{\bar{R}_b}}\left[\frac{\partial\bar{h}}{\partial\bar{x}}\right]+\bar{v}_s\left[\frac{\partial\bar{h}}{\partial\bar{y}}\right]_s-\Gamma\frac{1}{\bar{\rho}_s}\left[\frac{\bar{u}_s}{1+\bar{\delta}/\bar{R}_b}\left(\frac{\partial\bar{p}}{\partial\bar{x}}\right)_s+\bar{v}_s\left(\frac{\partial\bar{p}}{\partial\bar{y}}\right)_s\right]=0 \qquad (38)$$

where the derivatives of the enthalpy can be explicitly written in terms of the other unknowns and their derivatives when the gas model is specified. The concentration derivatives at the shock are provided by the rate Equation 16 assuming the flow is frozen with respect to dissociation through the shock. This is a reasonable assumption since the shock wave thickness is very small compared to the dissociational relaxation length. Under this assumption the concentration derivatives are:

$$\frac{\partial c_i}{\partial\bar{y}}=\frac{\bar{\omega}_{i_s}\left[1+\bar{\delta}/\bar{R}_b\right]}{\bar{\rho}_s\left[(1+\bar{\delta}/\bar{R}_b)\bar{v}_s-\bar{u}_s\frac{d\bar{\delta}}{d\bar{x}}\right]} \qquad (39)$$

Equations 33 through 38 constitute a set of simultaneous algebraic equations in the derivatives $\frac{\partial\bar{\rho}}{\partial\bar{y}}$, $\frac{\partial\bar{p}}{\partial\bar{y}}$, $\frac{\partial\bar{u}}{\partial\bar{y}}$ and $\frac{\partial c_i}{\partial\bar{y}}$ evaluated at the shock. The coefficients of Equation 33 may now be found and the interpolation polynomials are formulated.

The final equation necessary for solution of the problem is the following geometric relation:

$$\frac{d\bar{\delta}}{d\bar{x}}=\left[1+\frac{\bar{\delta}}{\bar{R}_b}\right]\tan(\sigma-\theta) \qquad (40)$$

This relation is derived from the geometry of the shock layer.

The boundary conditions must be specified to complete the problem description. On the body surface

$$\bar{v}_b=0 \qquad (41)$$

which insures that there is no mass flow across the body surface or, equivalently, that the body surface is a streamline.

The boundary conditions at the shock take the form of a set of equations in $\bar{v}_s$, $\bar{\rho}_s$, $\bar{p}_s$ and T_s. The gas is frozen with respect to dissociation through the shock as previously stated. It is assumed that the gas is in vibrational equilibrium immediately after passing through the shock. This leads to the solution of a set of simultaneous equations for the shock variables by means of an iterative scheme since the shock temperature T_s cannot be determined explicitly because of the form of the vibration term in the energy equation. The equations to be solved are:

$$\bar{\rho}_\infty \bar{U}_\infty \sin\sigma = \bar{\rho}_s \bar{q}_s \cos\psi \tag{42}$$

$$\bar{U}_\infty \cos\sigma = \bar{q}_s \sin\psi \tag{43}$$

$$\bar{p}_\infty + \frac{2\gamma_\infty}{\gamma_\infty - 1} \bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma = \bar{p}_s + \frac{2\gamma_\infty}{\gamma_\infty - 1} \bar{\rho}_s \bar{q}_s^2 \cos^2\psi \tag{44}$$

$$\bar{h}_s + \frac{\bar{q}_s^2}{2} = \bar{h}_\infty + \frac{\bar{U}_\infty^2}{2} \tag{45}$$

where the subscripts ∞ and s refer to free stream and post shock conditions and ψ is the angle of the post shock velocity with the shock normal. Figure 3 shows the geometry of the shock waves.

Equations 9, 14, 26, 27, 30, 31 and 40 are the equations which must be solved for the unknowns σ, $\bar{\delta}$, $\bar{p}_b$, $\bar{\rho}_b$, $\bar{u}_b$, c_{i_b} and T_b. For convenience the following list presents a summary of variables and their interdependence.

Unknown dependent variables

$\bar{\delta}(\bar{x})$, $\sigma(\bar{x})$, $\bar{u}_b(\bar{x})$, $\bar{p}_b(\bar{x})$, $\bar{\rho}_b(\bar{x})$, $c_{b_i}(\bar{x})$, $T_b(x)$

Given quantities

$\bar{p}_\infty$, $\bar{\rho}_\infty$, $\bar{U}_\infty$

$\bar{r}_b(\bar{x})$, $\bar{R}_b(\bar{x})$, $\theta_b(\bar{x})$.

Parameters whose functional dependence is known

$\bar{u}_s(\sigma)$, $\bar{v}_s(\sigma)$, $\bar{\rho}_s(\sigma)$, $\bar{p}_s(\sigma)$

from Equations 42 to 45.

Boundary conditions

$$\bar{u}_b(o) = 0 \qquad \sigma(o) = \frac{\pi}{2} \qquad \bar{\delta}(o) = \bar{\delta}_o$$

where $\bar{\delta}_o$ is an assumed value of shock stand-off distance which is initially unknown in the analysis.

The matching condition used to obtain the proper value of $\bar{\delta}_o$ takes the form of a singularity in Equation 22. This equation may be expanded into the following form:

$$A \left[\frac{d \ln \bar{\rho}_b}{d \ln \bar{u}_b} + 1 \right] \frac{d\bar{u}_b}{d\bar{x}} + B = 0 \tag{46}$$

where A and B are continuous functions of the dependent and independent variables. The x-momentum equation may be written for points on the body surface:

$$\bar{u}_b \frac{d\bar{u}_b}{d\bar{x}} = - \frac{\Gamma}{\bar{\rho}_b} \frac{d\bar{p}_b}{d\bar{x}} \tag{47}$$

and

$$d \ln \bar{u}_b = - \Gamma \frac{d\bar{p}_b}{d\bar{\rho}_b} \frac{d \ln \bar{\rho}_b}{\bar{u}_b^2} \quad . \tag{48}$$

Therefore

$$\frac{d \ln \bar{\rho}_b}{d \ln \bar{u}_b} = - \frac{\bar{u}_b^2}{\Gamma \left[\frac{d\bar{p}_b}{d\bar{\rho}_b}\right]} \tag{49}$$

substituting this in Equation 46

$$A \left[1 - \frac{\bar{u}_b^2}{\Gamma \left(\frac{d\bar{p}_b}{d\bar{\rho}_b} \right)} \right] \frac{d\bar{u}_b}{d\bar{x}} + B = 0 \qquad (50)$$

The singularity used by Dorodnitsyn and Belotserkovskii is apparent in Equation 50. When $\bar{u}_b^2 = \Gamma \frac{d\bar{p}_b}{d\bar{\rho}_b}$, $\frac{d\bar{u}_b}{d\bar{x}}$ becomes infinite unless B also vanishes. The simultaneous vanishing of both coefficients of Equation 50 insures that the derivative is finite. This condition is met by varying the shock stand-off distance $\bar{\delta}_o$.

The point at which $\bar{u}_b^2 = \Gamma \frac{d\bar{p}_b}{d\bar{\rho}_b}$ is usually referred to as the sonic point. For equilibrium and frozen flows this corresponds to the equilibrium and frozen sound speeds since the flow is isentropic along the body surface. In non-equilibrium flow the thermodynamic processes are non-isentropic and the singular point on the body can no longer be taken as the sonic point.

One of the disadvantages of the one strip shock layer approach is that all details of the internal flow field are lost. A technique giving streamlines and other flow properties has been formulated recently in reference 21. The N strip approach, using linear interpolation polynomials, provides as much detail as is required on the strip boundaries by increasing N which provides a smaller grid size.

Several problems arise when the higher-order strip theories are used. The one strip method is a simple two point boundary value problem in which the initially unknown parameter is determined by the regularity condition on the velocity derivative at the sonic point on the body surface. If two or more strips are used, the problem is complicated by the occurence of singularities in the velocity derivatives along the strip boundaries as well as on the body. This again is a type of two point problem but the initially unknown parameter at the strip interface is the velocity on the stagnation streamline. If a two strip shock is assumed, the initial velocity midway between the shock and the body on the stagnation streamline must be assumed as well as the initial shock stand off distance.

The one strip shock layer with a second-order interpolation polynomial requires the matching condition on the body surface only while providing better accuracy than the true first-order approach. This technique is applied to a sphere in the following chapter.

THE SECOND-ORDER INTEGRAL METHOD APPLIED TO A SPHERE USING A ONE STRIP SHOCK

The simplest geometric configuration that can be used to demonstrate the practicality of the developments of the previous chapter is a sphere. Hence, in the illustrative example worked out in this paper is a spherical body surface is assumed. The fore portion of many re-entry vehicles are nearly spherical so the results of the example should be of practical interest.

The coordinate system used is the same as that outlined previously with the exception that the origin is now at the center of the body and the x coordinate along the body is replaced by the polar angle ϑ as shown in Figure 4. Conversion from the system of Chapter II to the spherical system can be made by noting that:

$$
\begin{aligned}
&\frac{d\theta}{dx} = -\frac{d\vartheta}{dx} = -\frac{1}{R_b} \\
&\frac{d}{dx} = \frac{1}{R_b}\frac{d}{d\vartheta} \\
&r_b = R_b \sin\vartheta \\
&\frac{dr_b}{dx} = \cos\vartheta \\
&y' = y + R_b \\
&L = R_b \quad .
\end{aligned}
\tag{51}
$$

The non-dimensional distance $\bar{x}$ is the same as ϑ and note that derivatives with respect to y' are the same as with respect to y since R_b is constant. The body radius of curvature R_b is taken to be one foot for simplicity.

The integral equations necessary for this example are Equations 26 and 27. In the coordinate system of Figure 3 they become:

$$\frac{d}{d\vartheta}\int_0^{\bar\delta} \bar r\bar\rho\bar u d\bar y - [\bar r_s\bar\rho_s\bar u_s]\frac{d\bar\delta}{d\vartheta} + [1+\bar\delta][\bar r_s\bar\rho_s\bar v_s] = 0 \qquad (52)$$

$$\frac{d}{d\vartheta}\int_0^{\bar\delta} \bar r\bar\rho\bar u\bar v d\bar y + [1+\bar\delta][\bar r_s(\Gamma\bar p_s + \bar\rho_s\bar v_s^2)] - \bar r_b\,\Gamma\bar p_b - \bar r_s\bar\rho_s\bar u_s\bar v_s\frac{d\bar\delta}{d\vartheta}$$

$$-\int_0^{\bar\delta} [\bar r\,(\Gamma\bar p + \bar\rho\bar u^2) + \Gamma\,(1+\bar y)\cos\theta\,\bar p]\,d\bar y = 0 \qquad (53)$$

The specific integrands which must be approximated by polynomials are:

$$\bar r\bar\rho\bar u,\ \bar r\bar\rho\bar u\bar v,\ [\Gamma\bar p + \bar\rho\bar u^2]\,\bar r,\ (1+\bar y)\,\bar p\ . \qquad (54)$$

Each of the first three of these functions can be approximated by means of the coefficients of the polynomial expansion as determined by the known conditions on the body and the shock and the first derivative on the shock. The coefficients of the expansion $(1 + \bar y)\bar p$ are obtained by using the conditions on the body and the shock along with the first derivative condition on the body. The reason for using the normal derivative at the body is that it can be found readily from Euler's Equation 7. On the body

$$\left(\frac{\partial\bar p}{\partial\bar y}\right)_b = \frac{\bar\rho_b\bar u_b^2}{\Gamma} \qquad (55)$$

This is a much simpler expression than the normal derivative at the shock.

If the necessary integrations indicated in Equation 52 and 53 are carried out, the following forms are obtained:

$$\frac{1}{3}\frac{d}{d\vartheta}\left\{\bar{\delta}[2(1+\bar{\delta})\bar{\rho}_s\bar{u}_s\sin\vartheta + \bar{\rho}_b\bar{u}_b\sin\vartheta - \frac{\bar{\delta}}{2}(\frac{\partial\bar{r}\bar{\rho}\bar{u}}{\partial\bar{y}})_s]\right\} - (1+\bar{\delta})\sin\vartheta\ \bar{\rho}_s\bar{u}_s\frac{d\bar{\delta}}{d\vartheta}$$
$$+ (1+\bar{\delta})^2\ \bar{\rho}_s\bar{v}_s\ \sin\vartheta = 0 \tag{56}$$

$$\frac{1}{3}\frac{d}{d\vartheta}\left\{\bar{\delta}\ \sin\vartheta\ [\frac{4+3\bar{\delta}}{2}\ \bar{\rho}_s\bar{u}_s\bar{v}_s - \frac{\bar{\delta}(1+\bar{\delta})}{2}\ (\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}})_s]\right\} + \frac{\bar{\delta}^2(1+\bar{\delta})}{6}\ \sin\vartheta\frac{d}{d\vartheta}[\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}}]_s$$
$$- \frac{\bar{\delta}\ \sin\vartheta}{2}\ \bar{\rho}_s\bar{u}_s\bar{v}_s\ \frac{d\bar{\delta}}{d\vartheta} - \frac{\bar{\delta}\ \cos\vartheta}{3}\ [\frac{4+3\bar{\delta}}{2}\ \bar{\rho}_s\bar{u}_s\bar{v}_s - \frac{\bar{\delta}(1+\bar{\delta})}{2}\ (\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}})_s]$$
$$- \frac{\sin\vartheta}{3}\ \frac{d\bar{\delta}}{d\vartheta}[\frac{4+3\bar{\delta}}{2}\ \bar{\rho}_s\bar{u}_s\bar{v}_s - \frac{\bar{\delta}(1+\bar{\delta})}{2}\ (\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}})_s] - \frac{\bar{\delta}\ \sin\vartheta}{3}\ \frac{(4+3\bar{\delta})}{2}\ \frac{d}{d\vartheta}(\bar{\rho}_s\bar{u}_s\bar{v}_s)$$
$$+ \frac{\bar{\delta}\ \sin\vartheta}{6}\ (1+2\bar{\delta})\ \frac{d\bar{\delta}}{d\vartheta}\ (\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}})_s = 0 \tag{57}$$

The derivatives with respect to $\bar{y}$ in Equations 56 and 57 must be evaluated from the general equations of motion specified to the shock. The derivatives of the shock variables with respect to ϑ can be obtained from the solutions to the oblique shock equations. These expressions will be derived later.

The continuity Equation 56 must now be expanded and combined with the ϑ-momentum Equation 47 to explicitly exhibit the looping or singular condition on the body surface. If Equations 56 and 47 are combined, the following form results:

$$\bar{\rho}_b\bar{\delta}\ \sin\vartheta\,[1 - \frac{\bar{u}_b^2}{\Gamma\ \frac{d\bar{p}_b}{d\bar{\rho}_b}}\ ]\ \frac{d\bar{u}_b}{d\vartheta} = -\ \bar{\delta}\bar{\rho}_b\bar{u}_b\cos\vartheta\ - 3(1+\bar{\delta})^2\ \bar{\rho}_s\bar{v}_s\ \sin\vartheta$$
$$-2(1+\bar{\delta})\bar{\delta}\bar{\rho}_s\bar{u}_s\cos\vartheta\ - 2\bar{\delta}(1+\bar{\delta})\sin\vartheta\frac{d(\bar{\rho}_s\bar{u}_s)}{d\vartheta}\ - \sin\vartheta\frac{d\bar{\delta}}{d\vartheta}[\bar{\rho}_b\bar{u}_b - (1+\bar{\delta})\bar{\rho}_s\bar{u}_s$$
$$+2\bar{\delta}\bar{\rho}_s\bar{u}_s] + \bar{\delta}\ \frac{d\bar{\delta}}{d\vartheta}\ [\frac{\partial\ \ (1+\bar{\delta})\sin\vartheta\ \bar{\rho}\bar{u}}{\partial\bar{y}}]_s + \frac{\bar{\delta}^2}{2}\ \frac{d}{d\vartheta}\ [\frac{\partial\ \ (1+\bar{\delta})\sin\vartheta\ \bar{\rho}\bar{u}}{\partial\bar{y}}]_s\ . \tag{58}$$

The entire right hand side of this equation must vanish when the condition:

$$\bar{u}_b^2 = \Gamma \frac{d\bar{p}_b}{d\bar{\rho}_b} \tag{59}$$

is satisfied. This condition is fulfilled when the proper shock stand-off distance is used.

The gas model to be used must be chosen before any of the $\bar{y}$ derivatives in the preceeding equations can be evaluated. For simplicity the gas is taken as a simple diatomic gas following Lighthill (18). In the Lighthill model it is assumed that the energy stored in the vibrational degrees of freedom is one half of the fully excited classical value. This simplifies the analysis in that the vibrational energy is now a linear function of the temperature rather than a complicated expression similar to Equation 13. The vibrational energy is then taken to be:

$$e_v = \frac{R_2 T}{2} \quad . \tag{60}$$

The enthalpy of the gas is

$$h = (4+\alpha)\, R_2 T + \alpha D \quad . \tag{61}$$

The energy equation for a Lighthill gas may be written:

$$H_T = h + \frac{u^2+v^2}{2} = (4+\alpha)\, R_2 T + \alpha\, D + \frac{u^2+v^2}{2} \quad . \tag{62}$$

Equation 62 is non-dimensionalized by using the substitutions given in Equation 21 to obtain:

$$\frac{1}{2} = (4+\alpha)\bar{T} + \alpha\, \bar{D} + \frac{\bar{u}^2+\bar{v}^2}{2} \quad . \tag{63}$$

Since the gas model considered is a pure diatomic gas, only two chemical reactions are needed:

$$A_2 + A \underset{K_r^{(1)}}{\overset{K_d^{(1)}}{\rightleftharpoons}} 2A + A$$

$$A_2 + A_2 \quad \underset{K_r^{(2)}}{\overset{K_d^{(2)}}{\rightleftharpoons}} \quad 2A + A_2 \tag{64}$$

where A_2 denotes the molecular species. Both chemical reactions contribute to the source function given in Equation 17.

Let α be the mass fraction of the atomic species. The species continuity or rate equation may be written

$$\frac{u}{1 + \frac{y}{R_b}} \frac{\partial \alpha}{\partial x} + v \frac{\partial \alpha}{\partial y} = K_d^{(1)} \frac{\rho\alpha(1-\alpha)}{\mu_A} - 2K_r^{(1)} \frac{\rho^2\alpha^3}{\mu_A^2} + K_d^{(2)} \frac{\rho(1-\alpha)^2}{2\mu_A} - K_R^{(2)} \frac{\rho^2\alpha^2(1-\alpha)}{\mu_A^2} \tag{65}$$

where the superscript (1) refers to the first reaction and (2) to the second. The forward and reverse reaction coefficients are denoted by K_d and K_r and the molecular weight of the atomic species is given by μ_A. This rate equation is now specialized to the present example by applying it along the body surface.

$$u_b \frac{\partial \alpha_b}{\partial x} = K_d^{(1)} \rho_b \frac{\alpha_b(1-\alpha_b)}{\mu_A} - 2K_r^{(1)} \frac{\rho_b^2\alpha_b^3}{\mu_A^2} + K_d^{(2)} \frac{\rho_b(1-\alpha_b)^2}{2\mu_A} - K_r^{(2)} \frac{\rho_b^2\alpha_b^2(1-\alpha_b)}{\mu_A^2} \tag{66}$$

A more meaningful form for nonequilibrium may be obtained by nondimensionalizing and introducing a nonequilibrium parameter. Let

$$\beta = \frac{K_r^{(2)}}{K_r^{(1)}}$$

$$K_E = \frac{K_d^{(i)}}{K_r^{(i)}}$$

$$\bar{K}_r^{(1)} = \frac{\rho_{\infty_T}^2 R_b K_r^{(1)}}{q_{\infty_M} \mu_A^2}$$

$$\bar{K}_E = \frac{K_E\, \mu_A}{2\rho_{\infty_T}}$$

$$\bar{\lambda} = \bar{K}_E\,(1-\alpha) - \bar{\rho}\alpha^2 \tag{67}$$

using these substitutions the rate equation reads

$$\bar{u}_b \frac{d\alpha_b}{d\vartheta} = \bar{\rho}_b\, \bar{K}_r^{(1)} \bar{\lambda}\ \ [2\alpha_b + \beta\,(1-\alpha_b)] \ . \tag{68}$$

The parameter $\bar{\lambda}$ is a measure of the magnitude of the departure of the gas from thermodynamic equilibrium. For equilibrium flow $\bar{\lambda}$ is identically zero while for nonequilibrium conditions $\bar{\lambda}$ takes on a non-zero value which depends on how far the gas is out of equilibrium.

The rate and energy equations for the assumed gas model have been derived and the $\bar{y}$ derivatives of the dependent variables at the shock wave surface must now be obtained. The equations of motion are specified to the shock in the following form:

Continuity

$$(1+\bar{\delta})\bar{\rho}_s \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)_s - \bar{\rho}_s \frac{d\bar{\delta}}{d\vartheta}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_s + [(1+\bar{\delta})\bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}][\frac{\partial \bar{\rho}}{\partial \bar{y}}]_s = \Delta_s \ , \tag{69}$$

x-momentum

$$\bar{\rho}_s\ [(1+\bar{\delta})\bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}][\frac{\partial \bar{u}}{\partial \bar{y}}]_s - \Gamma \frac{d\bar{\delta}}{d\vartheta}\,[\frac{\partial \bar{p}}{\partial \bar{y}}]_s = \square_s \ , \tag{70}$$

y momentum

$$\bar{\rho}_s\ [(1+\bar{\delta})\bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}][\frac{\partial \bar{v}}{\partial \bar{y}}]_s + \Gamma\,(1+\bar{\delta})\ [\frac{\partial \bar{p}}{\partial \bar{y}}]_s = \nabla_s \ , \tag{71}$$

and energy

$$\bar{\rho}_s^2\ \bar{u}_s[\frac{\partial \bar{u}}{\partial \bar{y}}]_s + \bar{\rho}_s^2\ \bar{v}_s[\frac{\partial \bar{v}}{\partial \bar{y}}]_s - 4\Gamma\ \bar{p}_s\ [\frac{\partial \bar{\rho}}{\partial \bar{y}}]_s + 4\Gamma\ \bar{\rho}_s\ [\frac{\partial \bar{p}}{\partial \bar{y}}]_s = \Phi_s \tag{72}$$

where

$$\Delta_s = -\left[\frac{d\bar{\rho}\bar{u}}{d\vartheta}\right]_s - 2\,\bar{\rho}_s\bar{v}_s - \bar{\rho}_s\bar{u}_s\,\frac{\cos\vartheta}{\sin\vartheta}$$

$$\square_s = -\,\bar{\rho}_s\bar{u}_s\bar{v}_s - \Gamma\left[\frac{d\bar{p}}{d\vartheta}\right]_s - \bar{\rho}_s\bar{u}_s\left[\frac{d\bar{u}}{d\vartheta}\right]_s$$

$$\nabla_s = \bar{\rho}_s\bar{u}_s^2 - \bar{\rho}_s\bar{u}_s\left[\frac{d\bar{v}}{d\vartheta}\right]_s$$

$$\Phi_s = \bar{\rho}_s^2\left[3\Gamma\,\frac{\bar{p}_s}{\bar{\rho}_s} - \bar{D}\right]\left[\frac{\partial\alpha}{\partial\bar{y}}\right]_s$$

$$\left[\frac{\partial\alpha}{\partial\bar{y}}\right]_s = \frac{\beta\,\bar{K}_E\quad\bar{K}_r^{(1)}\,\bar{\rho}_s\,(1+\bar{\delta})}{(1+\bar{\delta})\,\bar{v}_s - \bar{u}_s\,\frac{d\bar{\delta}}{d\vartheta}}\quad. \tag{73}$$

Equations 69 through 72 are used to evaluate the $\bar{y}$ derivatives of $\bar{u}$, $\bar{v}$, $\bar{p}$ and $\bar{\rho}$ at the shock. The solutions are given in Appendix A. Angular derivatives of these $\bar{y}$ forms are also required and are given in Appendix A.

The angular derivatives of the shock variables are required to obtain solutions for the normal derivatives in the above system. The shock variables and their derivatives are provided by solution of the Rankine-Hugoniot equations. The equations that must be satisfied across the shock are

$$\bar{\rho}_\infty\,\bar{U}_\infty\,\sin\sigma = \bar{\rho}_s\,\bar{q}_s\,\cos\psi \tag{74}$$

$$\bar{U}_\infty\,\cos\sigma = \bar{q}_s\,\sin\psi \tag{75}$$

$$\Gamma\,\bar{p}_\infty + \bar{\rho}_\infty\,\bar{U}_\infty^2\,\sin^2\sigma = \Gamma\,\bar{p}_s + \bar{\rho}_s\,\bar{q}_s^2\,\cos^2\psi \tag{76}$$

$$7/2\,\bar{T}_\infty + \frac{\bar{U}_\infty^2}{2} = (4+\alpha)\,\bar{T}_s + \alpha\,\bar{D} + \frac{\bar{q}_s^2}{2}\quad. \tag{77}$$

The solutions to this system are given in Appendix B as functions of the shock angle σ which is one of the unknowns in the problem.

The formulation of the one strip shock case is completed by the equation of state:

$$\bar{p} = (1+\alpha)\ \bar{\rho}\bar{T} \tag{78}$$

where $\bar{T} = T/T_{\infty_t}$ and the geometric equation

$$\frac{d\bar{\delta}}{d\vartheta} = -\ (1+\bar{\delta})\ \cot\ (\sigma + \vartheta)\quad . \tag{79}$$

The following list presents a summary of the dependent variables and refers to the equation providing a solution for each.

unknown dependent variable	obtained from equation
$\bar{u}_b$	C1
σ	C2
$\bar{p}_b$	30
α_b	68
$\bar{\rho}_b$	63
$\bar{T}_b$	78
$\bar{\delta}$	79

boundary conditions

$$\sigma(o) = \pi/2 \qquad \bar{u}_b(o) = 0 \qquad \bar{\delta}(o) = \bar{\delta}_o$$

The equations referred to in Appendix C are the final forms of the continuity and y-momentum equations. The y-momentum equation is second-order in σ and provides a solution for σ. The continuity equation as previously discussed provides a solution for $\bar{u}_b$.

The general procedure followed in obtaining the required solution is to assume an initial value of the shock stand-off distance $\bar{\delta}_o$. Using this value of $\bar{\delta}_o$, the equations of motion are integrated from the shock to the body along the axis of symmetry. This provides the initial values of the dependent variables at the stagnation point. The governing differential equations are then integrated along the body starting from the axis of symmetry. This integration proceeds until Equation 59 is satisfied or the right hand side of Equation 58 vanishes. If both of these conditions are not satisfied simultaneously, a new value of $\bar{\delta}_o$ is chosen and the procedure is repeated until continuity of the velocity derivative on the body surface is guaranteed.

NUMERICAL PROGRAMS AND RESULTS

The analytical formulation of the nonequilibrium blunt body problem of Chapter III must be put in suitable form for numerical computation. The complexity of the problem dictates that the numerical computations be divided into two sections. First, the fluid properties along the axis of symmetry must be known to initiate the integration required in the problem and second, solutions of the differential equations which provide body conditions and shock shape must be obtained.

The Stagnation Streamline

The proper values of the fluid properties at the stagnation point on the body surface must be known for a given shock stand-off distance before a complete solution of the problem can be realized. These values must be obtained by solving a set of simultaneous differential and algebraic equations along the stagnation streamline. On the $\vartheta = 0$ streamline, the velocity component along the body surface vanishes. This condition applied to Equations 65, 7 and 63 leads to Equations 80, 81 and 83. Equations 82 and 84 are just the regular polynomial for pressure and the equation of state. Partial derivatives with respect to $\bar{y}$ are replaced by totals along the axis of symmetry since the dependent variables are only functions of $\bar{y}$. The required equations are the rate equation:

$$\frac{d\alpha}{d\bar{y}} = \frac{\bar{\rho}\,\bar{K}_r^{(1)}\bar{\lambda}}{\bar{v}}\,[2\alpha + \beta\,(1 - \alpha)] \quad , \tag{80}$$

Euler's equation

$$\frac{d\bar{v}}{d\bar{y}} = -\frac{\Gamma}{\bar{\rho}\bar{v}}\,\frac{d\bar{p}}{d\bar{y}} \quad , \tag{81}$$

pressure polynomial

$$\bar{p} = \frac{1}{1 + \bar{y}} \left[a_o + a_1 \frac{\bar{y}}{\bar{\delta}_o} + a_2 \left(\frac{\bar{y}}{\bar{\delta}_o}\right)^2 + a_3 \left(\frac{\bar{y}}{\bar{\delta}_o}\right)^3\right] \quad , \tag{82}$$

energy equation

$$\bar{\rho} = \frac{\bar{p}\ \Gamma^{\frac{4+\alpha}{1+\alpha}}}{\frac{1}{2} - \alpha\bar{D} - \frac{\bar{T}^2}{2}} \quad , \tag{83}$$

equation of state

$$\bar{T} = \frac{\bar{p}}{(1+\alpha)\bar{\rho}} \quad . \tag{84}$$

The continuity equation applied on the stagnation streamline is not used in this system but is replaced by the pressure polynomial. The polynomial expansion is used everywhere in the shock layer due to the approximation of the pressure integrand of Equation 53 and therefore, it was decided to apply it to the axis of symmetry. The technique used by Shih _et al._ (21) differed from the method presented here in that the limiting form of the continuity equation was used in place of Equation 82. When the continuity equation is used, a polynomial expansion of the $\bar{\rho}\bar{u}$ product is required to obtain an expression for the $\frac{\partial \bar{u}}{\partial \bar{x}}$ which appears. This leads to a somewhat more complicated system of equations. Thus the pressure polynomial expansion is used instead.

The accuracy of the solution to the main problem depends directly on the accuracy of the solution to the stagnation streamline problem. Since an approximate expression is used for the pressure, it is imperative that the approximation be as good as possible. For this reason, the pressure expression is written as a cubic which is a higher order approximation than that used for $\bar{r}^j\bar{\rho}\bar{u}$ in Equation 33. One more boundary condition is required to obtain the next term and this condition is obtained from Euler's equation which requires that the normal derivative of the pressure at the body vanish since $\bar{v}$ vanishes. The boundary conditions are:

$$\bar{p}_{\bar{y}=0} = \bar{p}_b \qquad \bar{p}_{\bar{y}=\bar{\delta}_o} = \bar{p}_s$$

$$\left(\frac{d\bar{p}}{d\bar{y}}\right)_{\bar{y}=0} = 0 \qquad \left(\frac{d\bar{p}}{d\bar{y}}\right)_{\bar{y}=\bar{\delta}_o} = \left(\frac{d\bar{p}}{d\bar{y}}\right)_s \quad . \tag{85}$$

To satisfy the boundary conditions, the coefficients become:

$$a_o = \bar{p}_b \qquad a_1 = \bar{\delta}_o\ \bar{p}_b$$

$$a_2 = (3+2\ \bar{\delta}_o)(\bar{p}_s - \bar{p}_b) - \bar{\delta}_o\ (1+\bar{\delta}_o)\left(\frac{d\bar{p}}{d\bar{y}}\right)_s$$

$$a_3 = \bar{\delta}_o\ (1+\bar{\delta}_o)\left(\frac{d\bar{p}}{d\bar{y}}\right)_s - (2+\bar{\delta}_o)(\bar{p}_s - \bar{p}_b) \quad . \tag{86}$$

The technique used to solve Equations 80 to 84, with the a_i's given by 85 and 86, is based on assuming an initial shock stand-off distance $\bar{\delta}_o$ and body pressure $\bar{p}_b$. The differential equations are then integrated from the shock to the body. The assumed body pressure is corrected after each pass until the velocity vanishes when $\bar{y} = 0$. A close examination of the method indicates that the solution of the stagnation streamline problem depends heavily on the characteristics of the assumed polynomial expansion for the pressure across the shock layer. The pressure increases continuously from the shock to the body and available information (21, 17) indicates that the pressure curve is concave downward at all points on the axis of symmetry. Since the pressure curve is concave downward everywhere, the second derivative must be negative everywhere. If the second derivative is obtained from Equation 82 it takes the form:

$$\frac{d^2\bar{p}}{d\bar{y}^2} = \frac{2}{1+\bar{y}} \left[\frac{a_2}{\bar{\delta}_o^2} + \frac{3a_3}{\bar{\delta}_o^2}\left(\frac{\bar{y}}{\bar{\delta}_o}\right) - \frac{d\bar{p}}{d\bar{y}}\right] \quad . \tag{87}$$

This function must be negative at both the shock and body. At the body

$$\left[\frac{d^2\bar{p}}{d\bar{y}^2}\right]_{\bar{y}=0} = \frac{2a_2}{\bar{\delta}_o^2} < 0 \tag{88}$$

and therefore

$$(3+2\bar{\delta}_o)(\bar{p}_s - \bar{p}_b) - \bar{\delta}_o\,(1+\bar{\delta}_o)\left[\frac{d\bar{p}}{d\bar{y}}\right]_s < 0 \quad . \tag{89}$$

This equation provides a lower bound on the values of $\bar{p}_b$ that may be used as an initial assumption in attempting to solve the required systems of equations. In particular:

$$\bar{p}_b > \bar{p}_s - \frac{\bar{\delta}_o\,(1+\bar{\delta}_o)}{3+2\bar{\delta}_o}\left[\frac{d\bar{p}}{d\bar{y}}\right]_s \quad . \tag{90}$$

The same condition applied at the shock yields:

$$\left[\frac{d^2\bar{p}}{d\bar{y}^2}\right]_{\bar{y}=\bar{\delta}_o} = \frac{2}{1+\bar{\delta}_o}\left[\frac{a_2}{\bar{\delta}_o^2} + \frac{3a_2}{\bar{\delta}_o^2} - \left(\frac{d\bar{p}}{d\bar{y}}\right)_{\bar{y}\,=\,\bar{\delta}_o}\right] < 0 \tag{91}$$

or

$$a_2 + 3a_3 - \bar{\delta}_o^2\left[\frac{d\bar{p}}{d\bar{y}}\right]_s < 0 \quad . \tag{92}$$

Substituting from Equation 86 the following form results:

$$\bar{p}_b < \bar{p}_s - \frac{(2+\bar{\delta}_o)\,\bar{\delta}_o}{3+\bar{\delta}_o}\left(\frac{d\bar{p}}{d\bar{y}}\right)_s \quad . \tag{93}$$

Equations 90 and 93 provide upper and lower bounds on possible body pressures that may be assumed to obtain a solution on the axis of symmetry and yet satisfy the second derivative condition. It is interesting to note that the allowable range of $\bar{p}_b$ may be controlled by proper combinations of $\bar{\delta}_o$ and $(\frac{d\bar{p}}{d\bar{y}})_s$.

Table 1 lists the flight conditions and various physical constants used to compute a stagnation streamline solution for oxygen under the special conditions indicated. The free stream conditions and the gas model dictate the values of the post shock variables and their derivatives. For the Lighthill model, the normal derivative of $\bar{p}$ on the stagnation streamline at the shock is large for nonequilibrium flow. This is due to the assumption of frozen flow with respect to dissociation through the shock and the accompanying steep concentration gradients immediately following the shock. Since $[\frac{d\bar{p}}{d\bar{y}}]_s$ is fixed, the bound on $\bar{p}_b$ is now controlled by the assumed value of $\bar{\delta}_o$. For values of $\bar{\delta}_o$ which are known to be approximately correct (21), the allowed range of $\bar{p}_b$ is too small to allow a solution along the axis of symmetry. For these free stream conditions, a new model for the pressure expansion must be obtained. It is required that the representation satisfy the same four boundary conditions of Equation 85 and that the second derivative condition also be met.

Since the allowed range of body pressures is too restrictive to allow a solution along the stagnation streamline when a third-order polynomial is used, an elliptic approximation was used. The elliptic form was chosen because of its geometric simplicity and the fact that all required boundary conditions of Equation 85 can be satisfied. Temperature, density, pressure and degree of dissociation profiles are given in Figures 5, 6, 7 and 8 for both equilibrium and nonequilibrium flows. These solutions were obtained through an inverse process in which the integration proceeds from the shock to the body. A shock stand-off distance $\bar{\delta}_o$ and a body pressure are initially assumed in the analysis, and the body pressure is adjusted after each pass until the velocity vanishes when $\bar{y} = 0$. Note that the only alteration required in Equations 80 to 84 to obtain the equilibrium solution is that the rate Equation 80 is replaced by the equilibrium condition $\bar{\lambda} = 0$.

In both the equilibrium and nonequilibrium solutions the elliptic pressure form was used. It is apparent that the elliptic approximation is not satisfactory if Figure 6 is carefully examined. The equilibrium solution is really an isentropic compression of the gas from the shock to the body whereas the nonequilibrium solution gives rise to an entropy increase in the compression because of the finite reaction rates. One of the major results of an entropy increase in an adiabatic flow is the resulting loss of stagnation pressure. Since both flows are required to be in equilibrium at the stagnation point, the equilibrium stagnation pressure must be higher than the nonequilibrium pressure and the temperatures also follow the same trend. Figures 5 and 6 indicate the opposite effect is true. This is an unfortunate consequence of the elliptic approximation and more directly of the domination of nonequilibrium pressure expansion by the first derivative of the pressure at the shock.

An interesting method for determining the stagnation point properties in nonequilibrium flow may be derived if one knows the normal derivative of $\bar{v}$ at the stagnation point. If the rate Equation 80 is differentiated with respect to $\bar{y}$ and applied to the stagnation point, the following is obtained:

$$[\frac{d\alpha}{d\bar{y}}]_o[\frac{d\bar{v}}{d\bar{y}}]_o = [\bar{\rho}\ \bar{K}_r^{(1)}]_o[2\alpha + \beta\ (1-\alpha)]_o[\frac{d\bar{\lambda}}{d\bar{y}}]_o \tag{94}$$

where

$$\bar{\lambda} = \bar{K}_E\ (1-\alpha) - \bar{\rho}\ \alpha^2 \tag{95}$$

and

$$[\frac{d\bar{\lambda}}{d\bar{y}}]_o = -[\frac{(1-\alpha)}{(1+\alpha)}\ T\ \frac{d\bar{K}_E}{dT} + \bar{K}_E + 2\alpha\bar{\rho}]_o[\frac{d\alpha}{d\bar{y}}]_o - [\alpha^2 + T\frac{d\bar{K}_E}{dT}\ \frac{(1-\alpha)}{\bar{\rho}}]_o[\frac{d\bar{\rho}}{d\bar{y}}]_o\ . \tag{96}$$

Substituting Equation 96 into 94:

$$\left\{[\frac{d\bar{v}}{d\bar{y}}]_o + \bar{\rho}\bar{K}_r^{(1)}\ [2\alpha + \beta(1-\alpha)][\frac{1-\alpha}{1+\alpha}\ T\ \frac{d\bar{K}_E}{dT} + \bar{K}_E + 2\alpha\bar{\rho}]\right\}_o [\frac{d\alpha}{d\bar{y}}]_o$$

$$+ [\bar{\rho}\bar{K}_r^{(1)}]_o[2\alpha + \beta\ (1-\alpha)][\alpha^2 - \frac{(1-\alpha)}{\bar{\rho}}\ T\ \frac{d\bar{K}_E}{dT}]_o[\frac{d\bar{\rho}}{d\bar{y}}]_o = 0\ . \tag{97}$$

Equation 97 is a homogeneous algebraic equation in the two derivatives $[\frac{d\alpha}{d\bar{y}}]_o$ and $[\frac{d\bar{\rho}}{d\bar{y}}]_o$.

The energy equation may be written in the form:

$$\frac{1}{2} = \frac{\bar{v}^2}{2} + \alpha\ \bar{D}\ + \frac{4+\alpha}{1+\alpha}\ \frac{\Gamma\bar{p}}{\bar{\rho}}\ . \tag{98}$$

Differentiating with respect to y and applying the result to the stagnation point:

$$[\ \bar{D}\ - 3\ \frac{\Gamma\bar{p}}{\bar{\rho}\ (1+\alpha)^2}\ ]_o\ [\frac{d\alpha}{d\bar{y}}]_o - [\frac{4+\alpha}{1+\alpha}\ \frac{\Gamma\bar{p}}{\bar{\rho}^2}\ ]_o\ [\frac{d\bar{\rho}}{d\bar{y}}\ ]_o = 0\ . \tag{99}$$

Equations 97 and 99 constitute a pair of homogeneous algebraic equations in the derivatives $[\frac{d\alpha}{d\bar{y}}]_o$ and $[\frac{d\bar{\rho}}{d\bar{y}}]_o$. If a non-trivial solution to this system exists, the determinant of the coefficients must vanish. This condition yields:

$$[\frac{\Gamma\bar{p}}{\bar{\rho}^2}(\frac{4+\alpha}{1+\alpha})]_o \left\{[\frac{d\bar{v}}{d\bar{y}}]_o + \bar{\rho}\bar{K}_r^{(1)}[2\alpha + \beta(1-\alpha)][\frac{1-\alpha}{1+\alpha} T \frac{d\bar{K}_E}{dT} + \bar{K}_E + 2\alpha\bar{\rho}]\right\}_o$$

$$+ [\bar{\rho}\bar{K}_r^{(1)}]_o[2\alpha + \beta(1-\alpha)]_o[\alpha^2 + \frac{(1-\alpha)}{\rho} T \frac{d\bar{K}_E}{dT}]_o[\bar{D} - \frac{3\Gamma\bar{p}}{\bar{\rho}(1+\alpha)^2}]_o = 0. \quad (100)$$

All conditions at the stagnation point now can be determined providing $[\frac{d\bar{v}}{d\bar{y}}]_o$ is known. The unknowns are $\bar{p}$, $\bar{\rho}$, α and T and the governing algebraic system is comprised of the equilibrium condition $\tilde{\lambda} = 0$, the energy Equation 98, the equation of state 84 and the constraint given by Equation 100. The derivative $[\frac{d\bar{v}}{d\bar{y}}]_o$ is determined from the assumed pressure expansion across the shock layer by using Euler's equation. Differentiating 81 and applying the result to the stagnation point:

$$[\frac{d\bar{v}}{d\bar{y}}]_o = - [- \frac{\Gamma}{\bar{\rho}} \frac{d^2\bar{p}}{d\bar{y}^2}]_o^{\frac{1}{2}} . \quad (101)$$

A computer program for solving the algebraic system described above was written using both the polynomial and elliptic pressure expansion to determine $[\frac{d\bar{v}}{d\bar{y}}]_o$. The program was written by assuming an initial temperature and then calculating $\bar{p}$, $\bar{\rho}$ and α from Equations 95, 98 and 84. These values of $\bar{p}$, $\bar{\rho}$, α and T were then substituted into Equation 100 to see if the assumed T was in fact a zero. A corrected value of T was then used and through successive passes, a solution was obtained.

The zero's of Equation 100 depend on the value of $[\frac{d\bar{v}}{d\bar{y}}]_o$ which in turn depends on $[\frac{d^2\bar{p}}{d\bar{y}^2}]_o$. The second derivative of the pressure also depends on the body pressure due to the approximate representation in the shock layer. For both pressure expansions, the digital computer program would iterate until values of body temperature and pressure were low enough that the second derivative of $\bar{p}$ would change signs. Reference to Equations

88 and 89 indicate this possibility for the polynomial while this behavior in the elliptic case means that the pressure expansion has jumped from the upper half which gives the true expansion to the lower half of the pressure ellipse. This is equivalent to reducing the body pressure to a value somewhat less than the shock pressure. This possibility for the elliptic pressure curve is presented because the body pressure is computed independent of position on the ellipse. The first derivative has a fixed sign and when the body pressure assumes a value less than the shock pressure, the approximation moves from the first to the third quadrant. This causes the change in sign of the second derivative.

The maximum value of the second derivative of the pressure on the body would be zero since the stagnation pressure is the maximum value in the shock layer. A solution setting the second derivative equal to zero was obtained but is not included here. The solution is independent of the shock layer pressure variation under these conditions. The values of the stagnation point variables obtained by integrating the nonequilibrium differential equations and by the second or algebraic scheme do not agree as is expected. The algebraic technique does show promise in that the results of the two solutions are the same order of magnitude at the stagnation point. At present, the integration method would necessarily be used to determine the stagnation point thermodynamic variables, because the algebraic method is not in a usable form since more research on the pressure second derivative is required.

Main Program

The numerical aspects of the differential system giving the required solution to the hypersonic blunt body problem were not fully investigated since the problem was terminated with the stagnation streamline. The basic idea used in solving the differential equations as set down previously in Chapter III is to assume a shock stand-off distance $\bar{\delta}_o$, integrate along the stagnation streamline to obtain starting conditions at the stagnation point and then integrate along the body surface until the matching condition (Equation 59) is satisfied or until the numerator or denominator associated with the matching condition in the continuity equation vanishes. A new value of $\bar{\delta}_o$ is assumed and the above process is repeated until continuity of the velocity derivative is guaranteed. The details and numerical problems associated with this technique are discussed in reference 14.

The method used to obtain solutions for the first-order approximation in reference 21 is a modification of the formal iteration procedure given above. Solutions of the differential equations along the body are obtained for upper and lower bounds on shock stand-off distance $\bar{\delta}_o$. A logarithmic search procedure is initiated using the stored upper and lower bound solutions. This technique eliminates the necessity of solving the stagnation streamline portion and a good part of the body surface part of the problem. The results obtained seem to indicate success in using the technique. For further information reference 14 provides a more complete discussion of the details.

Examination of either the continuity Equation C1 or the $\bar{y}$-momentum Equation C2 reveals the occurence of removable singularities on the axis of symmetry. These singularities prevent the initial derivatives in the numerical integration program from being obtained directly from the differential equations. A limiting form for the necessary derivatives at the stagnation point was obtained by assuming $[\frac{d\sigma}{d\vartheta}]_o = -1$. This is a reasonable assumption since the shock and fore portion of a blunt body have nearly the same shape in the vicinity of the axis of symmetry. Using this assumption and taking the limiting form of Equations G1 and G2:

$$[\frac{d^2\sigma}{d\vartheta^2}]_o = 1 + \frac{1}{2(1-\epsilon_o)}[\frac{d^2\epsilon}{d\sigma^2}]_o - \frac{\bar{\rho}_\infty \bar{U}_\infty^2 (1-\epsilon_o)}{2\Gamma\epsilon_o (\bar{p}_s - \bar{p}_\infty)} - \frac{(1+\bar{\delta}_o)(\bar{\delta}_o + 3)\bar{p}_s}{2\bar{\delta}_o^2 (\bar{p}_s - \bar{p}_\infty)}$$

$$+ \frac{\bar{\rho}_\infty \bar{U}_\infty^2}{2\Gamma(\bar{p}_s - \bar{p}_\infty)}[\frac{4+3\bar{\delta}_o}{\bar{\delta}_o} - \frac{3(1+\bar{\delta}_o)^2\epsilon_o}{\bar{\delta}_o^2} - (1+\bar{\delta}_o)^3(8\Gamma\bar{p}_s\bar{\rho}_\infty\bar{U}_\infty(1-\epsilon_o) + \epsilon_o\bar{U}_\infty(1+\bar{\delta}_o)\Phi_o)]$$

$$+ \frac{\bar{p}_b}{2(\bar{p}_s - \bar{p}_\infty)\bar{\delta}_o^2}(3 + 4\bar{\delta}_o + 4\bar{\delta}_o^2) \tag{102}$$

$$[\frac{d\sigma}{d\vartheta}]_o = -1 \tag{103}$$

$$[\frac{d\bar{u}_b}{d\vartheta}]_o = \frac{\Phi_o \bar{\delta}_o \bar{U}_\infty (1+\bar{\delta}_o)}{2\bar{\rho}_b [4\Gamma\bar{p}_s - 3\bar{\rho}_\infty\epsilon_o\bar{U}_\infty^2]} - \frac{3\bar{\delta}_o\bar{\rho}_\infty^2\bar{U}_\infty^3(1-\epsilon_o)}{\epsilon_o\bar{\rho}_b[4\Gamma\bar{p}_s - 3\bar{\rho}_\infty\epsilon_o\bar{U}_\infty^2]}$$

$$+ \frac{\bar{\delta}_o}{2\bar{\rho}_b\epsilon_o\bar{U}_\infty}[\epsilon_o\bar{\rho}_\infty\bar{U}_\infty(1-\epsilon_o) + \Gamma(\bar{p}_s - \bar{p}_\infty)(2(\frac{d^2\sigma}{d\vartheta^2})_o - [2 + \frac{1}{1-\epsilon}(\frac{d^2\epsilon}{d\sigma^2})_o])]$$

$$- \frac{\bar{\rho}_\infty\bar{U}_\infty}{\bar{\rho}_b}[\frac{4+3\bar{\delta}_o}{2\epsilon_o} + \frac{3}{2}\frac{(1+\bar{\delta}_o)^2}{\bar{\delta}_o}] \quad . \tag{104}$$

Notice that the second derivative of σ is required since the governing differential equation is second-order. Because of the symmetry of the problem, the shock angle σ and the velocity along the body surface $\bar{u}_b$ are the only dependent variables that possess non-zero first derivatives at the stagnation point. The initial derivatives given above along with the stagnation values of the dependent variables are sufficient to provide the starting conditions for the integration of the governing differential equation.

RECOMMENDATIONS FOR FURTHER STUDY

The next step in continuing this problem must be more work on the stagnation streamline. The discussion of the results shown in Figure 6 clearly indicates that a new technique is needed for determining conditions along the axis of symmetry. A solution using a linear approximation for the pressure should be computed and this should be compared against both the elliptic solution and the results of reference 21.

The difficulty encountered with the axis of symmetry is the large value of the pressure derivative at the shock. A different or more realistic gas model may reduce this pressure derivative to a value which would allow the general polynomial expansion for pressure to be used. If diffusion effects are included using the present gas model, the same result may be obtained although current literature would not support this as the diffusion effect would be small. A considerable amount of effort could be expended on determining the properties along the stagnation streamline and at the stagnation point. Much remains to be done in this area for real gases.

The main program for determining shock shape and body flow parameter variation must be programmed and results obtained. This is a rather difficult problem as is evidenced by the complicated forms given in the appendix. The debugging of the main program may take a full year or even longer. The results obtained from the solution to the blunt body problem must be compared with those of reference 21. A comparison to determine which method is actually better is impossible at present since only the first-order linear solution has been completed. To draw any meaningful conclusions, a comparison must be made with both the linear one strip and two strip shock solutions.

The discovery of an algebraic method for determining the stagnation point properties may be the most significant result of this investigation. This technique as described in the previous chapter requires that the normal derivative of the velocity at the stagnation point be known. This normal derivative was obtained using the approximate pressure expansions used in the solution of the stagnation streamline problem. This method proved to be unsatisfactory as previously discussed.

An exact solution for the stagnation point properties using the algebraic method necessarily requires that a solution of the general equations of motion for the velocity derivative be obtained. Solving the equations of motion for the normal derivative of the velocity at the stagnation point is a difficult problem because some of the governing differential equations contain no useful information when they are applied along the body surface. The x-momentum equation is an example of this behavior.

If an exact solution for the normal derivative of the velocity at the stagnation point cannot be obtained, another technique may be of value. The pressure variation in the shock layer and along the body surface is only slightly affected by the nonequilibrium state of the flow (15, 21). Since the required derivative of velocity depends directly on the second derivative of the pressure at the stagnation point, consideration should be given to the idea of solving for the normal derivative of the velocity using perfect gas. This assumes that the pressure curves are of the same shape in both the nonequilibrium and perfect gas cases. Any result obtained using this technique should be of value.

All computer programs used in this investigation are on file in the Aerospace Engineering Department and should be consulted for any information concerning numerical techniques or required programs.

ACKNOWLEDGMENTS

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Table 1. Atmospheric and thermodynamic data

Free Stream Conditions (19)	Coefficients for Oxygen (11)
Altitude = 150,000 ft.	$\bar{K}_E = \frac{755103}{T} e^{-\frac{106884}{T}}$
$p_\infty = 2.725$ psf	$\bar{K}_r^{(1)} = 130911$
$U_\infty = 15{,}000$ ft/sec	$\beta(T) = \frac{3085.71}{T}$
$t_\infty = 480.7\ ^\circ R$	$\bar{D} = .715412$

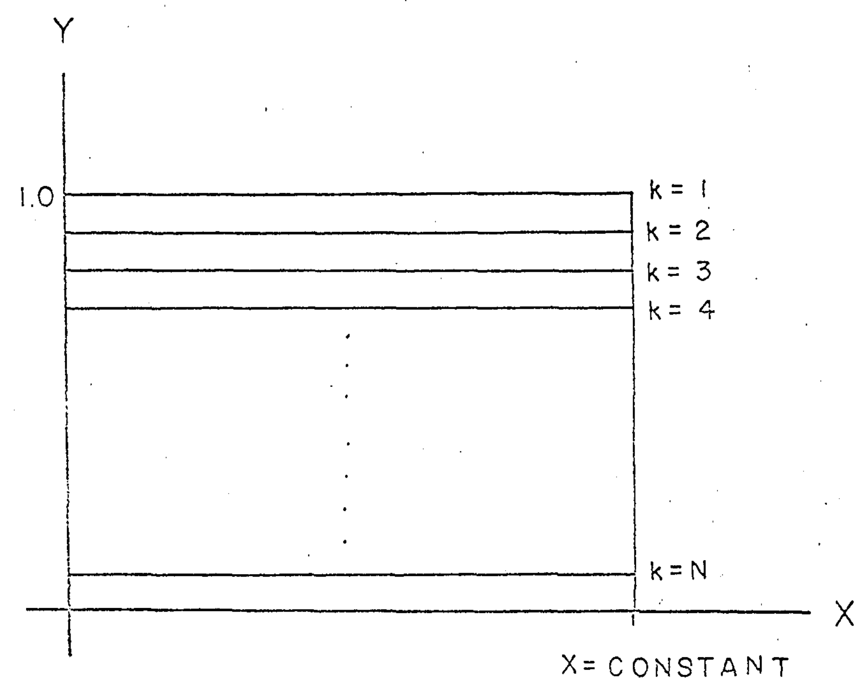

Figure 1. Rectangular domain

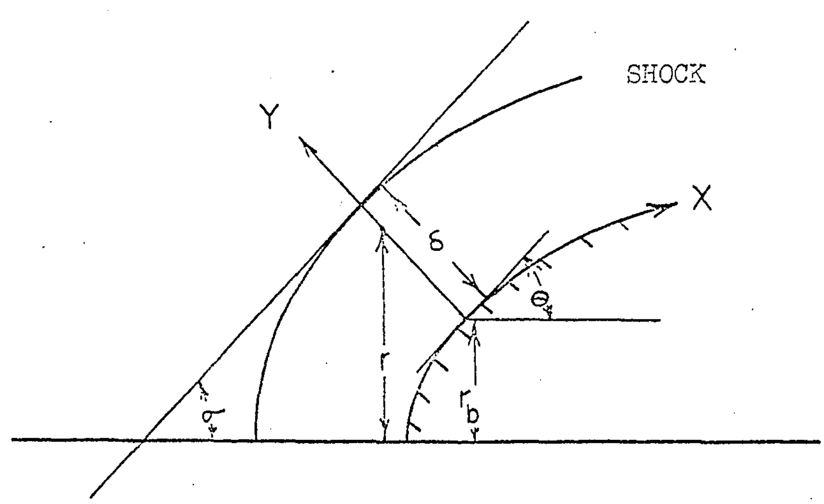

Figure 2. Co-ordinate system

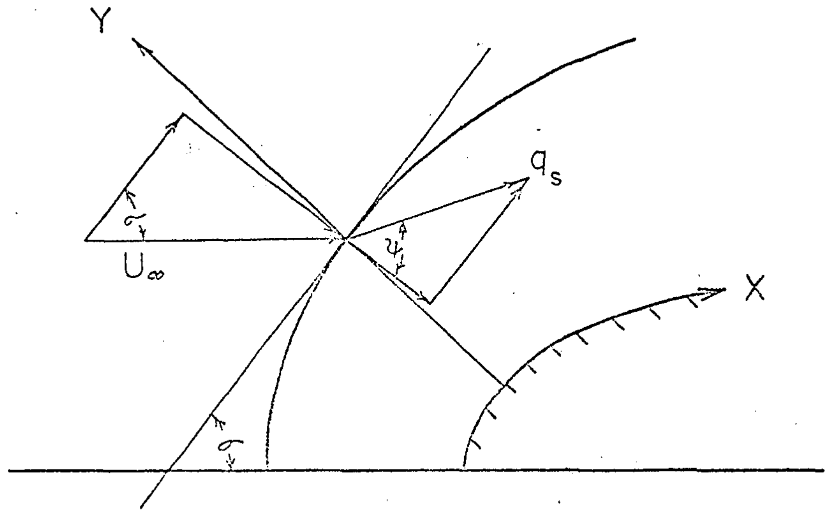

Figure 3. Shock wave geometry

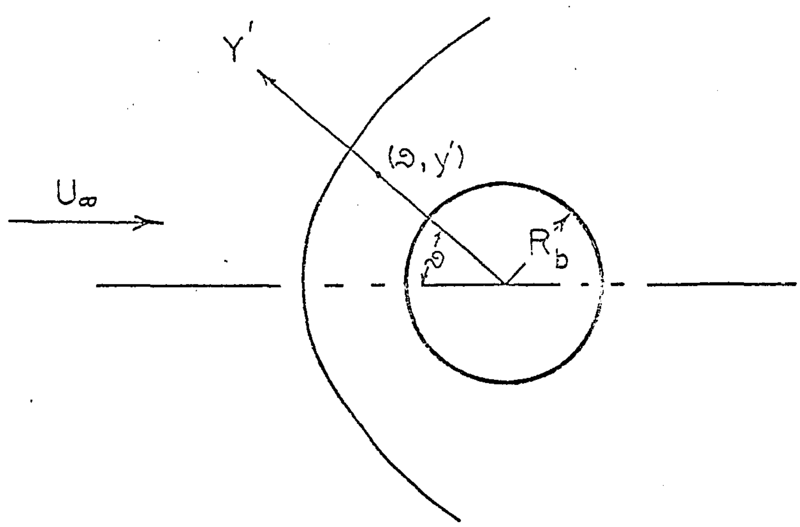

Figure 4. Spherical body

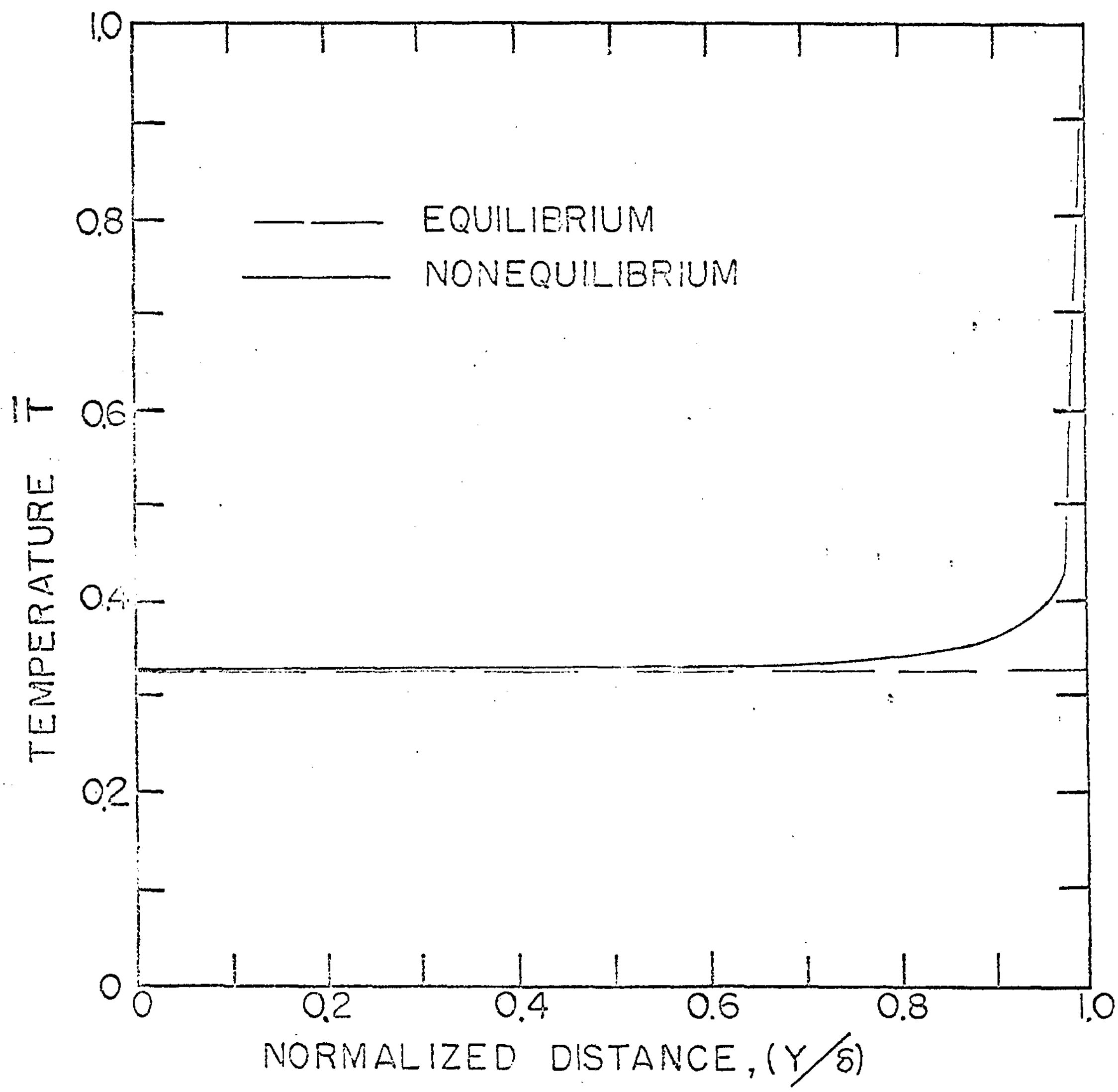

Figure 5. Stagnation streamline temperature profile

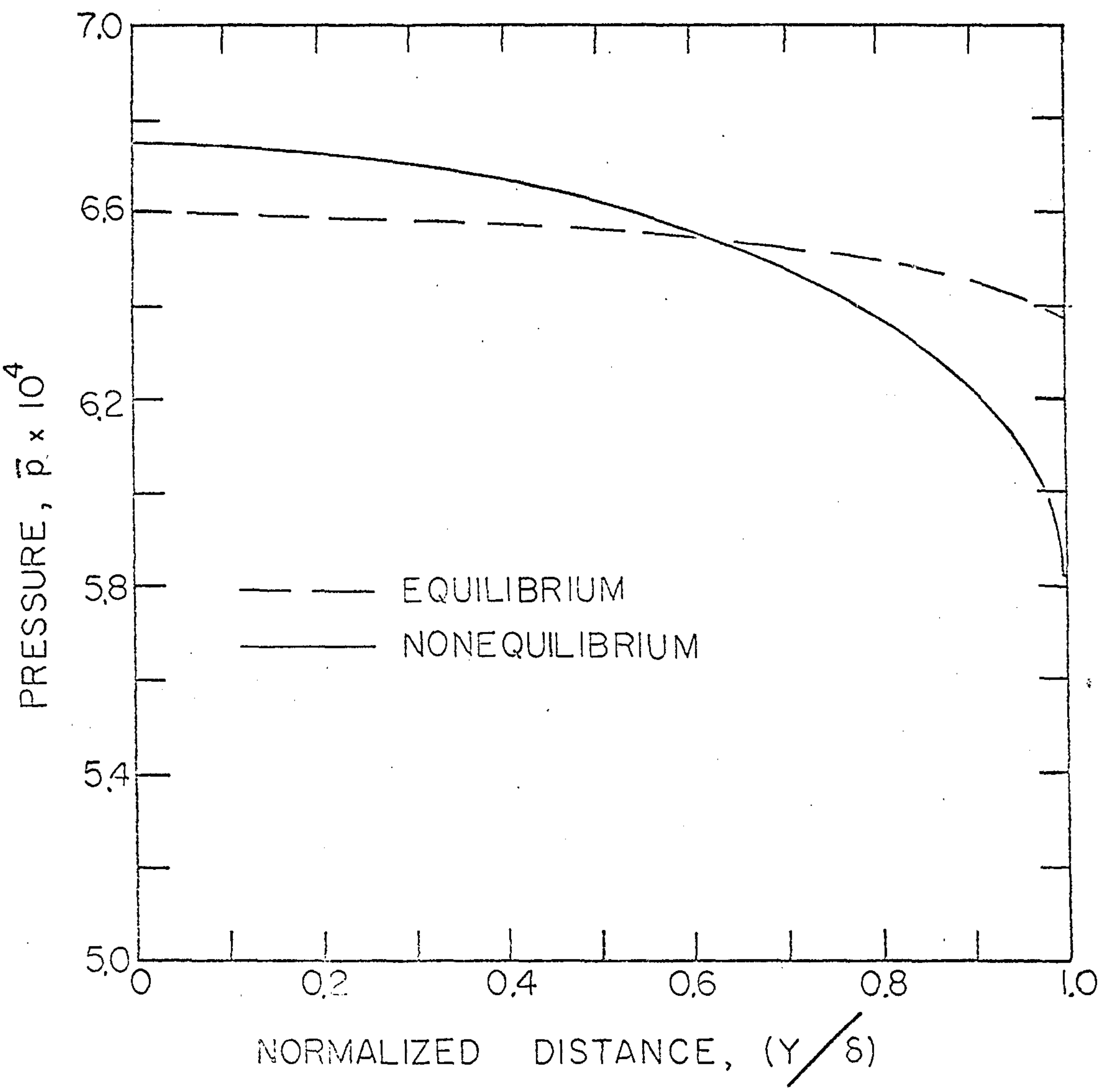

Figure 6. Stagnation streamline pressure variation

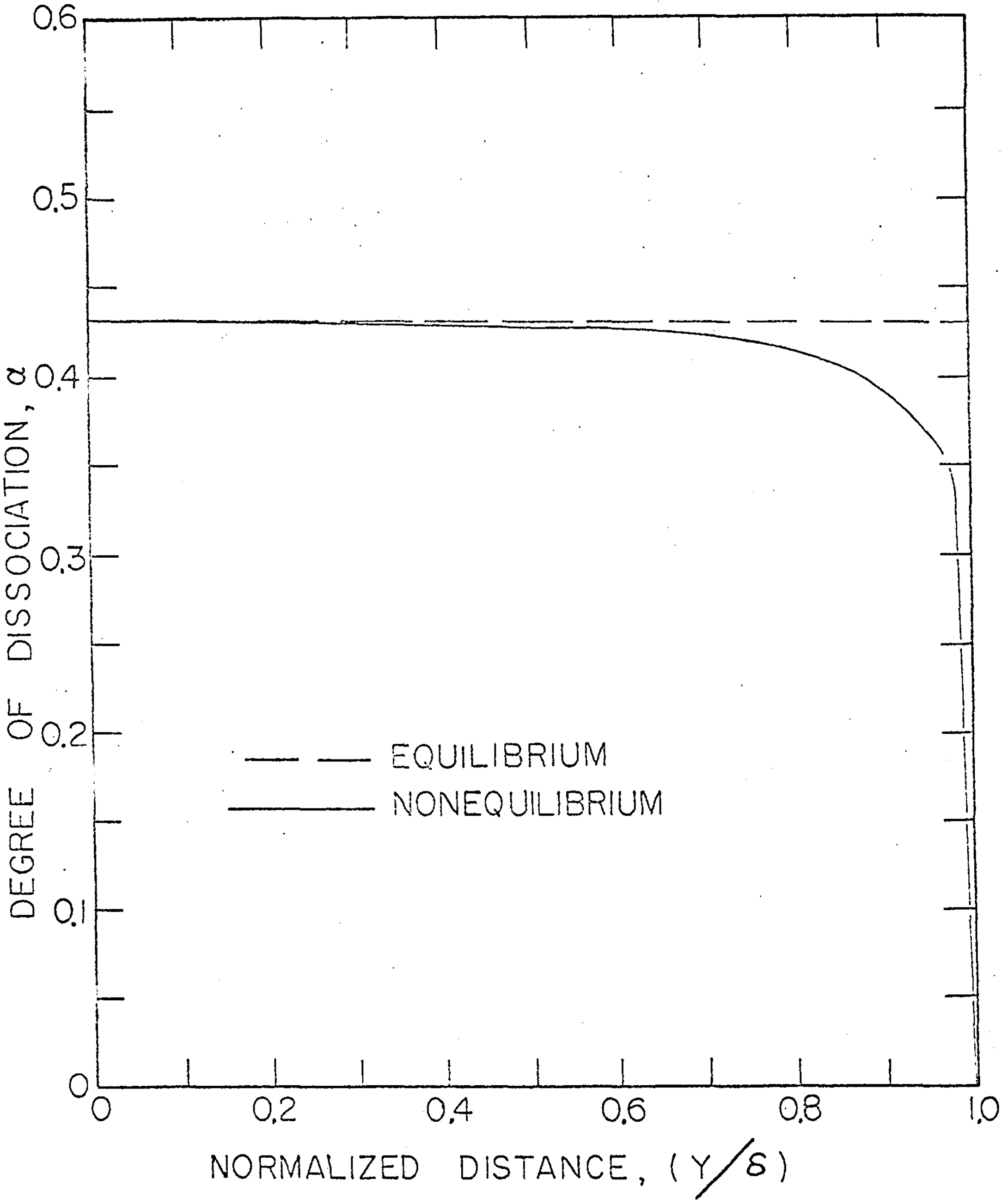

Figure 7. Oxygen concentration along the stagnation streamline

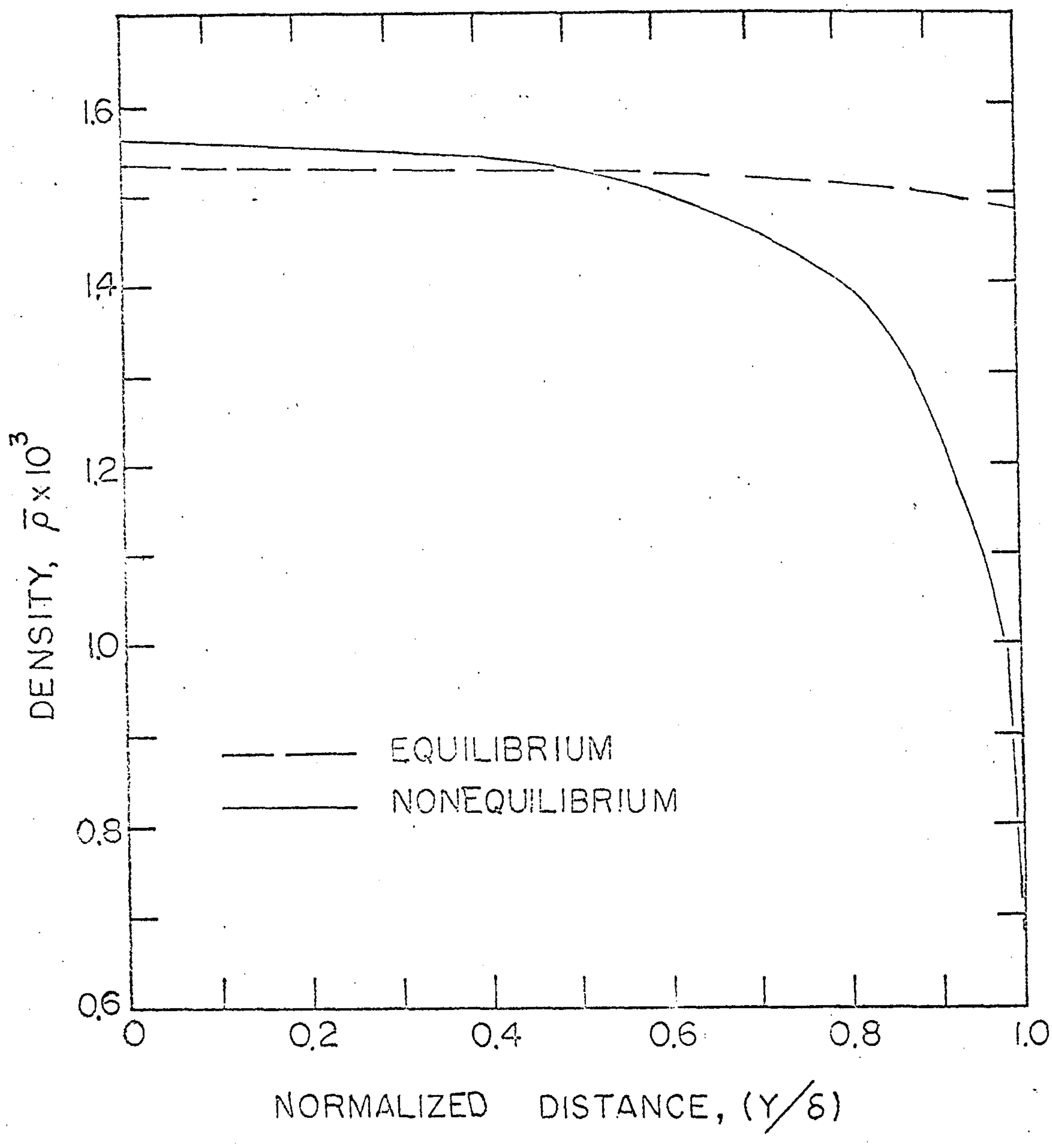

Figure 8. Density profile

APPENDIX A

Normal Derivatives of the Dependent Variables and Angular Derivatives Evaluated at the Shock

The system of equations that must be solved for the normal derivatives at the shock are given as Equations 69-72. They are repeated here for convenience.

$$(1+\bar{\delta})\bar{\rho}_s \left[\frac{\partial \bar{v}}{\partial \bar{y}}\right]_s - \bar{\rho}_s \frac{d\bar{\delta}}{d\vartheta}\left[\frac{\partial \bar{u}}{\partial \bar{y}}\right]_s + \left[(1+\bar{\delta})\bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}\right]\left[\frac{\partial \bar{\rho}}{\partial \bar{y}}\right]_s = \Delta_s \tag{A1}$$

$$\bar{\rho}_s\left[(1+\bar{\delta})\bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}\right]\left[\frac{\partial \bar{u}}{d\bar{y}}\right]_s - \Gamma \frac{d\bar{\delta}}{d\vartheta}\left[\frac{\partial \bar{p}}{\partial \bar{y}}\right]_s = \square_s \tag{A2}$$

$$\bar{\rho}_s\left[(1+\bar{\delta})\ \bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta}\right]\left[\frac{\partial \bar{v}}{\partial \bar{y}}\right]_s + \Gamma(1+\bar{\delta})\left[\frac{\partial \bar{p}}{\partial \bar{y}}\right]_s = \nabla_s \tag{A3}$$

$$\bar{\rho}_s^2\ \bar{v}_s\left[\frac{\partial \bar{v}}{\partial \bar{y}}\right]_s + \bar{\rho}_s^2 \bar{u}_s\left[\frac{\partial \bar{u}}{\partial \bar{y}}\right]_s - 4\Gamma\ \bar{p}_s\left[\frac{\partial \bar{\rho}}{\partial \bar{y}}\right]_s + 4\Gamma \bar{\rho}_s\left[\frac{\partial \bar{p}}{\partial \bar{y}}\right]_s = \Phi_s \tag{A4}$$

where the right hand sides are defined in Equation 73. Let Δ represent the determinant of the coefficients, and

$$c = (1+\bar{\delta})\ \bar{v}_s - \bar{u}_s \frac{d\bar{\delta}}{d\vartheta} \tag{A5}$$

then:

$$\Delta = \begin{vmatrix} (1+\bar{\delta})\ \bar{\rho}_s & -\bar{\rho}_s \frac{d\bar{\delta}}{d\vartheta} & c & 0 \\ 0 & \bar{\rho}_s\ c & 0 & -\Gamma \frac{d\bar{\delta}}{d\vartheta} \\ \bar{\rho}_s\ c & 0 & 0 & \Gamma(1+\bar{\delta}) \\ \bar{\rho}_s^2\ \bar{v}_s & \bar{\rho}_s^2\ \bar{u}_s & -4\Gamma\ \bar{p}_s & 4\Gamma\ \bar{\rho}_s \end{vmatrix}$$

or

$$\Delta = \bar{\rho}_s^2\ \Gamma c \left\{4\bar{p}_s \Gamma\left[(1+\bar{\delta})^2 + \left(\frac{d\bar{\delta}}{d\vartheta}\right)^2\right] - 3\bar{\rho}_s\ c^2\right\}. \tag{A6}$$

Solutions for the normal derivatives at the shock are then:

$$\left[\frac{\partial\bar{v}}{\partial\bar{y}}\right]_s = \frac{4\blacktriangle_s\Gamma^2}{\Delta}(1+\bar{\delta})\,\bar{p}_s\,\bar{\rho}_s c + \frac{\Box_s}{\Delta}\Gamma\bar{\rho}_s\,(1+\bar{\delta})\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta} - \bar{\rho}_s\bar{u}_s c\right]$$

$$+\frac{\nabla_s}{\Delta}\bar{\rho}_s\Gamma\left[4\Gamma\bar{p}_s\left(\frac{d\bar{\delta}}{d\vartheta}\right)^2 - c\left(4\bar{\rho}_s c + \bar{\rho}_s\bar{u}_s\frac{d\bar{\delta}}{d\vartheta}\right)\right] + \frac{\Phi_s}{\Delta}\Gamma\bar{\rho}_s\,(1+\bar{\delta})c^2 \qquad \text{(A7)}$$

$$\left[\frac{\partial\bar{u}}{\partial\bar{y}}\right]_s = -\frac{4\blacktriangle_s\Gamma^2}{\Delta}\bar{p}_s\bar{\rho}_s c\frac{d\bar{\delta}}{d\vartheta} - \frac{\Phi_s}{\Delta}\Gamma\bar{\rho}_s c^2\frac{d\bar{\delta}}{d\vartheta} + \frac{\nabla_s}{\Delta}\Gamma\bar{\rho}_s\frac{d\bar{\delta}}{d\vartheta}\left[4\Gamma(1+\bar{\delta})\bar{p}_s + \bar{\rho}_s\bar{v}_s c\right]$$

$$+\frac{\Box_s}{\Delta}\Gamma\bar{\rho}_s\left[4\Gamma(1+\bar{\delta})^2\,\bar{p}_s - c\,\bar{\rho}_s\,(4c - (1+\bar{\delta})\,\bar{v}_s)\right] \qquad \text{(A8)}$$

$$\left[\frac{\partial\bar{\rho}}{\partial\bar{y}}\right]_s = \frac{\nabla_s}{\Delta}\bar{\rho}_s^3\Gamma\left\{(1+\bar{\delta})\left(\bar{u}_s\frac{d\bar{\delta}}{d\vartheta} + 4c\right) + \bar{v}_s\left(\frac{d\bar{\delta}}{d\vartheta}\right)^2\right\}$$

$$-\frac{\Box_s\Gamma}{\Delta}\bar{\rho}_s^3\left[\frac{d\bar{\delta}}{d\vartheta}(4c - (1+\bar{\delta})\,\bar{v}_s) - (1+\bar{\delta})^2\,\bar{u}_s\right]$$

$$-\frac{\blacktriangle_s\Gamma}{\Delta}\bar{\rho}_s^3\,c\left\{\bar{u}_s\frac{d\bar{\delta}}{d\vartheta} + 4c - (1+\bar{\delta})\,\bar{v}_s\right\}$$

$$-\frac{\Phi_s\Gamma}{\Delta}\bar{\rho}_s^2\,c\left[(1+\bar{\delta})^2 + \left(\frac{d\bar{\delta}}{d\vartheta}\right)^2\right] \qquad \text{(A9)}$$

$$\left[\frac{\partial\bar{p}}{\partial\bar{y}}\right]_s = \frac{\nabla_s}{\Delta}\bar{\rho}_s^2\,c\left[4\Gamma\,(1+\bar{\delta})\,\bar{p}_s + \bar{\rho}_s\,\bar{v}_s\,c\right] - \frac{\Box_s}{\Delta}\bar{\rho}_s^2\,c\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta} - \bar{\rho}_s\,\bar{u}_s\,c\right]$$

$$-\frac{\Phi_s}{\Delta}\bar{\rho}_s^2\,c^3 - \frac{4\blacktriangle_s}{\Delta}\Gamma\bar{p}_s\,\bar{\rho}_s^2\,c^2 \qquad \text{(A10)}$$

The continuity equation and $\bar{y}$-momentum equations require angular derivatives of the normal derivatives of $\bar{\rho}\bar{u}$ and $\bar{\rho}\bar{u}\bar{v}$ at the shock. The normal derivative of $\bar{\rho}\bar{u}$ may be written

$$\left[\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{y}}\right]_s = \frac{\Box_s}{\Delta}\Gamma(1+\bar{\delta})\bar{\rho}_s^2\left[4\Gamma(1+\bar{\delta})\bar{p}_s - 3\bar{\rho}_s\,(1+\bar{\delta})\,\bar{v}_s^2 + \bar{\rho}_s\bar{u}_s^2(1+\bar{\delta}) + 4\bar{\rho}_s\bar{u}_s\bar{v}_s\frac{d\bar{\delta}}{d\vartheta}\right]$$

$$+\frac{\nabla_s}{\Delta}\Gamma\bar{\rho}_s^2\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta} + \bar{\rho}_s\bar{v}_s^2\frac{d\bar{\delta}}{d\vartheta} + 4\bar{\rho}_s\bar{u}_s\bar{v}_s\,(1+\bar{\delta}) - 3\bar{\rho}_s\bar{u}_s^2\frac{d\bar{\delta}}{d\vartheta}\right](1+\bar{\delta})$$

$$-\frac{\blacktriangle_s}{\Delta}\bar{\rho}_s^2\,\Gamma c\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta} + 3\bar{\rho}_s\bar{u}_s\,c\right]$$

$$-\frac{\Phi_s}{\Delta}\Gamma\bar{\rho}_s^2\,c(1+\bar{\delta})\left[\bar{u}_s\,(1+\bar{\delta}) + \bar{v}_s\,d\bar{\delta}/d\vartheta\right] \qquad \text{(A11)}$$

Let

$$F = 4\bar{p}_s \, \Gamma \, [(1+\bar{\delta})^2 + (d\bar{\delta}/d\vartheta)^2] - 3\bar{\rho}_s c^2 \tag{A12}$$

then

$$\triangle = \bar{\rho}_s^2 \, \Gamma c F$$

The normal derivative of $\bar{\rho}\bar{u}$ may be written:

$$\begin{aligned}[\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{y}}]_s = {} & \frac{\Box_s(1+\bar{\delta})}{cF} \, [4\Gamma \, (1+\bar{\delta})\bar{p}_s - 3\bar{\rho}_s(1+\bar{\delta}) \, \bar{v}_s^2 + \bar{\rho}_s\bar{u}_s^2 \, (1+\bar{\delta}) + 4\bar{\rho}_s\bar{u}_s\bar{v}_s \, \frac{d\bar{\delta}}{d\vartheta}] \\ & + \frac{\nabla_s}{cF} \, (1+\bar{\delta}) \, [4\Gamma\bar{p}_s \, \frac{d\bar{\delta}}{d\vartheta} + \bar{\rho}_s\bar{v}_s^2 \, \frac{d\bar{\delta}}{d\vartheta} + 4\bar{\rho}_s\bar{u}_s\bar{v}_s(1+\bar{\delta}) - 3\bar{\rho}_s\bar{u}_s^2 \, \frac{d\bar{\delta}}{d\vartheta}] \\ & - \frac{\Delta_s}{F} \, [4\Gamma\bar{p}_s \, \frac{d\bar{\delta}}{d\vartheta} + 3\bar{\rho}_s \, \bar{u}_s \, c] \\ & - \frac{\Phi_s}{F}(1+\bar{\delta}) \, [\bar{u}_s \, (1+\bar{\delta}) + \bar{v}_s \, \frac{d\bar{\delta}}{d\vartheta}]\end{aligned} \tag{A13}$$

Let

$$A = (1+\delta) \, [4\Gamma \, (1+\bar{\delta}) \, \bar{p}_s - 3\bar{\rho}_s \, (1+\bar{\delta}) \, \bar{v}_s^2 + \bar{\rho}_s\bar{u}_s^2 \, (1+\bar{\delta}) + 4\bar{\rho}_s\bar{u}_s\bar{v}_s \, \frac{d\bar{\delta}}{d\vartheta}]$$

$$B = 4\Gamma \, \bar{p}_s \, \frac{d\bar{\delta}}{d\vartheta} + 3 \, \bar{\rho}_s\bar{u}_s \, c$$

$$D = (1+\bar{\delta}) \, [4\Gamma\bar{p}_s \, \frac{d\bar{\delta}}{d\vartheta} + \bar{\rho}_s\bar{v}_s^2 \, \frac{d\bar{\delta}}{d\vartheta} + 4\bar{\rho}_s\bar{u}_s\bar{v}_s(1+\bar{\delta}) - 3\bar{\rho}_s\bar{u}_s^2 \, \frac{d\bar{\delta}}{d\vartheta}]$$

$$E = (1+\bar{\delta}) \, [\bar{u}_s(1+\bar{\delta}) + \bar{v}_s \, \frac{d\bar{\delta}}{d\vartheta}]$$

Then

$$[\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{y}}]_s = \frac{\Box_s A}{cF} + \frac{\nabla_s D}{cF} - \frac{C \, \Delta_s B}{cF} - \frac{c \, \Phi_s E}{cF} \tag{A14}$$

and

$$\begin{aligned}\frac{d}{d\vartheta} [\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{y}}]_s = {} & \frac{1}{cF} \, [\Box_s \frac{dA}{d\vartheta} + A \, \frac{d\Box_s}{d\vartheta} + \nabla_s \frac{dD}{d\vartheta} + D \, \frac{d\nabla_s}{d\vartheta} - c\Delta_s \frac{dB}{d\vartheta} - cB \, \frac{d\Delta_s}{d\vartheta} - c\Phi_s \frac{dE}{d\vartheta} - cE \, \frac{d\Phi_s}{d\vartheta} \\ & - (\Delta_s B + \Phi_s E) \, \frac{dc}{d\vartheta}] - [\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{y}}]_s \, [\frac{1}{F} \, \frac{dF}{d\vartheta} + \frac{1}{c} \, \frac{dc}{d\vartheta}]\end{aligned} \tag{A15}$$

where $\dfrac{dc}{d\vartheta} = [(1+\bar{\delta}) \quad \dfrac{d\bar{v}_s}{d\sigma} - \dfrac{d\bar{\delta}}{d\vartheta} \, \dfrac{d\bar{u}_s}{d\sigma}] \, \dfrac{d\sigma}{d\vartheta} + \bar{v}_s \, \dfrac{d\bar{\delta}}{d\vartheta} - \bar{u}_s \, \dfrac{d^2\bar{\delta}}{d\vartheta^2}$

$$\frac{dF}{d\vartheta} = 8\bar{p}_s\Gamma\,\frac{d\bar{\delta}}{d\vartheta}\left[(1+\bar{\delta}) + \frac{d^2\bar{\delta}}{d\vartheta^2}\right] - 6\bar{p}_s c\,\frac{dc}{d\vartheta} + \left\{4\Gamma\left[(1+\bar{\delta})^2+\left(\frac{d\bar{\delta}}{d\vartheta}\right)^2\right]\frac{d\bar{p}_s}{d\sigma} - 3c^2\,\frac{d\bar{\rho}_s}{d\sigma}\right\}\frac{d\sigma}{d\vartheta}$$

$$\frac{dA}{d\vartheta} = \frac{A}{1+\bar{\delta}}\,\frac{d\bar{\delta}}{d\vartheta} + (1+\bar{\delta})\,\frac{d\bar{\delta}}{d\vartheta}\Big[4\Gamma\bar{p}_s - 3\bar{\rho}_s\bar{v}_s^2 + \bar{\rho}_s\bar{u}_s^2 + 4\,\frac{d\sigma}{d\vartheta}\Big(\bar{\rho}_s\bar{u}_s\,\frac{d\bar{v}_s}{d\sigma} + \bar{\rho}_s\bar{v}_s\,\frac{d\bar{u}_s}{d\sigma} + \bar{u}_s\bar{v}_s$$

$$\frac{d\bar{\rho}_s}{d\sigma}\Big)\Big] + (1+\bar{\delta})4\bar{\rho}_s\bar{u}_s\bar{v}_s\,\frac{d^2\bar{\delta}}{d\vartheta^2} + \frac{d\sigma}{d\vartheta}\Big[4\Gamma\,\frac{d\bar{p}_s}{d\sigma} - 3\bar{v}_s^2\,\frac{d\bar{\rho}_s}{d\sigma} - 6\bar{\rho}_s\bar{v}_s\,\frac{d\bar{v}_s}{d\sigma} + \bar{u}_s^2\,\frac{d\bar{\rho}_s}{d\sigma}$$

$$+ 2\bar{\rho}_s\bar{u}_s\,\frac{d\bar{u}_s}{d\sigma}\Big]\,(1+\bar{\delta})^2$$

$$\frac{dB}{d\vartheta} = 4\Gamma\bar{p}_s\,\frac{d^2\bar{\delta}}{d\vartheta^2} + \left[4\Gamma\,\frac{d\bar{\delta}}{d\vartheta}\,\frac{d\bar{p}_s}{d\sigma} + \left(3\bar{\rho}_s\,\frac{d\bar{u}_s}{d\sigma} + 3\bar{u}_s\,\frac{d\bar{\rho}_s}{d\sigma}\right)c\right]\frac{d\sigma}{d\vartheta} + 3\,\bar{\rho}_s\bar{u}_s\,\frac{dc}{d\vartheta}$$

$$\frac{dD}{d\vartheta} = \frac{D}{1+\bar{\delta}}\,\frac{d\bar{\delta}}{d\vartheta} + (1+\bar{\delta})\,\frac{d^2\bar{\delta}}{d\vartheta^2}\left[4\Gamma\bar{p}_s + \bar{\rho}_s\bar{v}_s^2 - 3\bar{\rho}_s\,\bar{u}_s^2\right]$$

$$+ \frac{d\bar{\delta}}{d\vartheta}\left\{4\bar{\rho}_s\bar{u}_s\bar{v}_s + \frac{d\sigma}{d\vartheta}\left[2\bar{\rho}_s\bar{v}_s\,\frac{d\bar{v}_s}{d\sigma} + \bar{v}_s^2\,\frac{d\bar{\rho}_s}{d\sigma} + 4\Gamma\,\frac{d\bar{p}_s}{d\sigma} - 6\bar{\rho}_s\bar{u}_s\,\frac{d\bar{u}_s}{d\sigma} - 3\bar{u}_s^2\,\frac{d\bar{\rho}_s}{d\sigma}\right]\right\}$$

$$(1+\bar{\delta}) + 4(1+\bar{\delta})^2\,\frac{d\sigma}{d\vartheta}\left[\bar{\rho}_s\bar{u}_s\,\frac{d\bar{v}_s}{d\sigma} + \bar{\rho}_s\bar{v}_s\,\frac{d\bar{u}_s}{d\sigma} + \bar{u}_s\bar{v}_s\,\frac{d\bar{\rho}_s}{d\sigma}\right]$$

$$\frac{dE}{d\vartheta} = \frac{E}{1+\delta}\,\frac{d\bar{\delta}}{d\vartheta} + (1+\bar{\delta})\,\bar{v}_s\,\frac{d^2\bar{\delta}}{d\vartheta^2} + (1+\bar{\delta})\bar{u}_s\,\frac{d\bar{\delta}}{d\vartheta} + \frac{d\sigma}{d\vartheta}(1+\bar{\delta})\left[(1+\bar{\delta})\frac{d\bar{u}_s}{d\sigma} + \frac{d\bar{\delta}}{d\vartheta}\,\frac{d\bar{v}_s}{d\sigma}\right]$$

$$\frac{d\nabla_s}{d\vartheta} = \bar{\rho}_s\bar{u}_s\left[\frac{d\bar{u}_s}{d\vartheta} - \frac{d^2\bar{v}_s}{d\vartheta^2}\right] + \left[\bar{u}_s - \frac{d\bar{v}_s}{d\vartheta}\right]\left[\bar{\rho}_s\,\frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\,\frac{d\bar{\rho}_s}{d\vartheta}\right]$$

$$\frac{d\square_s}{d\vartheta} = -\bar{\rho}_s\bar{u}_s\left[\frac{d\bar{v}_s}{d\vartheta} + \frac{d^2\bar{u}_s}{d\vartheta^2}\right] - \frac{d\bar{u}_s}{d\vartheta}\left[\bar{\rho}_s\bar{v}_s + \bar{\rho}_s\,\frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\,\frac{d\bar{\rho}_s}{d\vartheta}\right] - \bar{u}_s\bar{v}_s\,\frac{d\bar{\rho}_s}{d\vartheta} - \Gamma\,\frac{d^2\bar{p}_s}{d\vartheta^2}$$

$$\frac{d\Delta_s}{d\vartheta} = -\bar{\rho}_s\left[\frac{d^2\bar{u}_s}{d\vartheta^2} + 2\,\frac{d\bar{v}_s}{d\vartheta}\right] - \frac{2d\bar{\rho}_s}{d\vartheta}\left[\frac{d\bar{u}_s}{d\vartheta} + \bar{v}_s\right] - \frac{\cos\vartheta}{\sin\vartheta}\left[\bar{\rho}_s\,\frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\,\frac{d\bar{\rho}_s}{d\vartheta}\right]$$

$$- \bar{u}_s\,\frac{d^2\bar{\rho}_s}{d\vartheta^2} + \frac{\bar{\rho}_s\bar{u}_s}{\sin^2\vartheta}$$

$$\frac{d\Phi_s}{d\vartheta} = \frac{\Phi s}{c}\,\frac{dc}{d\vartheta} + \Phi_s\left[\frac{1}{1+\bar{\delta}}\,\frac{d\bar{\delta}}{d\vartheta} + \frac{1}{\bar{\rho}_s}\,\frac{d\bar{\rho}_s}{d\vartheta} + \frac{1}{\bar{K}^{(1)}_{r_s}}\,\frac{d\bar{K}^{(1)}_{r_s}}{d\vartheta} + \frac{1}{\beta_s}\,\frac{d\beta_s}{d\vartheta} + \frac{1}{\bar{K}_E}\,\frac{d\bar{K}_E}{d\vartheta}\right]$$

$$+ \Phi_s\left[\frac{2}{\bar{\rho}_s}\,\frac{d\bar{\rho}_s}{d\vartheta} - \frac{3\Gamma}{\bar{D} - 3\Gamma\,\frac{\bar{p}_s}{\bar{\rho}_s}}\left(\bar{\rho}_s\,\frac{d\bar{p}_s}{d\vartheta} - \bar{p}_s\,\frac{d\bar{\rho}_s}{d\vartheta}\right)\frac{1}{\bar{\rho}_s^2}\right]$$

$$\frac{d^2\bar{\delta}}{d\vartheta^2} = \frac{(1+\bar{\delta})}{\sin^2(\sigma+\vartheta)}\left(1+\frac{d\sigma}{d\vartheta}\right) - \frac{\cos(\sigma+\vartheta)}{\sin(\sigma+\vartheta)}\frac{d\bar{\delta}}{d\vartheta}$$

Similarly:

$$\frac{d}{d\vartheta}\left[\frac{\partial\bar{\rho}\bar{u}\bar{v}}{\partial\bar{y}}\right]_s = \left[\left(\frac{\partial\bar{v}}{\partial\bar{y}}\right)_s \frac{d(\bar{\rho}_s\bar{u}_s)}{d\sigma} + \left(\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{y}}\right)_s \frac{d\bar{v}_s}{d\sigma}\right]\frac{d\sigma}{d\vartheta} + \bar{\rho}_s\bar{u}_s\frac{d}{d\vartheta}\left[\frac{\partial\bar{v}}{\partial\bar{y}}\right]_s + \bar{v}_s\frac{d}{d\vartheta}\left[\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{y}}\right]_s$$

where
$$\frac{d}{d\vartheta}\left[\frac{\partial\bar{v}}{\partial\bar{y}}\right]_s = \left[\frac{\partial\bar{v}}{\partial\bar{y}}\right]_s\left[\frac{1}{F}\frac{dF}{d\vartheta} + \frac{1}{c}\frac{dc}{d\vartheta} + \frac{1}{\bar{\rho}_s}\frac{d\bar{\rho}_s}{d\vartheta}\right] + \frac{4(1+\bar{\delta})\bar{p}_s}{\bar{\rho}_s F}\frac{d\triangleright_s}{d\vartheta}$$

$$+ \frac{4\triangleright_s}{\bar{\rho}_s cF}\left[(1+\bar{\delta})\bar{p}_s\frac{dc}{d\vartheta} + (1+\bar{\delta})c\frac{d\bar{p}_s}{d\vartheta} + \bar{p}_s c\frac{d\bar{\delta}}{d\vartheta}\right]$$

$$+ \frac{\square_s(1+\bar{\delta})}{\bar{\rho}_s cF}\left[4\Gamma\bar{p}_s\frac{d^2\bar{\delta}}{d\vartheta^2} + 4\Gamma\frac{d\bar{p}_s}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta} - \bar{\rho}_s\bar{u}_s\frac{dc}{d\vartheta} - \bar{\rho}_s c\frac{d\bar{u}_s}{d\vartheta} - \bar{u}_s c\frac{d\bar{\rho}_s}{d\vartheta}\right]$$

$$+ \frac{1}{\bar{\rho}_s cF}\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta} - \bar{\rho}_s\bar{u}_s c\right]\left[\square_s\frac{d\bar{\delta}}{d\vartheta} + (1+\bar{\delta})\frac{d\square_s}{d\vartheta}\right] + \frac{(1+\bar{\delta})c}{\bar{\rho}_s F}\frac{d\Phi_s}{d\vartheta}$$

$$+ \frac{\Phi_s}{\bar{\rho}_s F}\left[2(1+\bar{\delta})\frac{dc}{d\vartheta} + \frac{c\,d\bar{\delta}}{d\vartheta}\right] - \frac{\nabla_s}{cF}\left[4c + \bar{u}_s\frac{d\bar{\delta}}{d\vartheta}\right]\frac{dc}{d\vartheta}$$

$$- \frac{\nabla}{\bar{\rho}_s cF}\Big[4\Gamma\frac{d\bar{p}_s}{d\vartheta}\left(\frac{d\bar{\delta}}{d\vartheta}\right)^2 + 8\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta}\frac{d^2\bar{\delta}}{d\vartheta^2} - c\Big(4\bar{\rho}_s\frac{dc}{d\vartheta} + 4c\frac{d\bar{\rho}_s}{d\vartheta} + \bar{\rho}_s\bar{u}_s\frac{d^2\bar{\delta}}{d\vartheta^2}$$

$$+ \bar{\rho}_s\frac{d\bar{u}_s}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta} + \bar{u}_s\frac{d\bar{\rho}_s}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta}\Big)\Big]$$

The indicated derivatives of the shock variables are presented in Appendix B.

APPENDIX B

Method of Obtaining Post Shock Conditions As Functions of the Shock Angle

The conditions which must be satisfied across the shock wave have been given in Chapter III and are repeated here for convenience.

$$\bar{\rho}_\infty \bar{U}_\infty \sin\sigma = \bar{\rho}_s \bar{q}_s \cos\psi \tag{B1}$$

$$\bar{U}_\infty \cos\sigma = \bar{q}_s \sin\psi \tag{B2}$$

$$\Gamma\bar{p}_\infty + \bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma = \Gamma\bar{p}_s + \bar{\rho}_s \bar{q}_s^2 \cos^2\psi \tag{B3}$$

$$7/2 \; \frac{\bar{p}_\infty}{\bar{\rho}_\infty} + \frac{\bar{U}_\infty}{2} = 4\Gamma \frac{\bar{p}_s}{\bar{\rho}_s} + \frac{\bar{q}_s^2}{2} \tag{B4}$$

Let

$$\epsilon = \bar{\rho}_\infty / \bar{\rho}_s$$

Then

$$\epsilon \bar{U}_\infty \sin\sigma = \bar{q}_s \cos\psi \tag{B5}$$

$$\bar{U}_\infty \cos\sigma = \bar{q}_s \sin\psi \tag{B6}$$

$$\Gamma\bar{p}_\infty + \bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma = \Gamma\bar{p}_s + \bar{\rho}_\infty/\epsilon \;\; \bar{q}_s^2 \cos^2\psi \tag{B7}$$

$$7/2 \; \Gamma \frac{\bar{p}_\infty}{\bar{\rho}_\infty} + \frac{\bar{U}_\infty^2}{2} = \frac{\bar{q}_s^2}{2} + 4\Gamma\epsilon \; \bar{p}_s/\bar{\rho}_s \tag{B8}$$

The normal deflection angle ψ can be eliminated from B7 by squaring Equation B5 and combining:

$$\Gamma\bar{p}_\infty + \bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma = \Gamma\bar{p}_s + \bar{\rho}_\infty \; \epsilon \, \bar{U}_\infty^2 \sin^2\sigma \tag{B9}$$

or

$$\bar{p}_s = \bar{p}_\infty + \frac{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma}{\Gamma} \; (1-\epsilon) \tag{B10}$$

From B5 and B6:

$$\bar{U}_\infty^2 \cos^2 \sigma + \epsilon^2 \bar{U}_\infty^2 \sin^2 \sigma = \bar{q}_s^2 \tag{B11}$$

Substituting this expression for $\bar{q}_s^2$ into B8 and solving for $\bar{p}_s$, one obtains:

$$\bar{p}_s = \frac{7\bar{p}_\infty}{8\epsilon} + \frac{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma}{8\Gamma\epsilon} - \epsilon \frac{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2 \sigma}{8\ \Gamma} \tag{B12}$$

If B10 and B12 are set equal, a quadratic in the density ratio is generated:

$$\epsilon^2 - \frac{8\epsilon}{7} \left[\frac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2 \sigma} + 1\right] + \frac{\bar{p}_\infty \Gamma}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2 \sigma} + \frac{1}{7} = 0 \tag{B13}$$

and

$$\epsilon = \frac{4}{7} \left[\frac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2 \sigma} + 1\right] - \left[\frac{16}{49} \left(\frac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma} + 1\right)^2 - \frac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma} - \frac{1}{7}\right]^{\frac{1}{2}} \tag{B14}$$

The density ratio is known and the shock pressure can be obtained from Equation B10. The temperature is obtained from the equation of state and the velocity components parallel and perpendicular to the local shock surface are obtained from Equations B5 and B6.

The normal and tangential components of velocity at the shock must be resolved into components along and normal to the local body surface. Figure 3 demonstrates the geometry of this problem.

The velocity immediately behind the shock can be obtained from B5 and B6 as:

$$\bar{q}_s = \bar{U}_\infty \left[\epsilon^2 \sin^2\sigma + \cos^2\sigma\right]^{\frac{1}{2}} \tag{B15}$$

From the geometry of Figure 3 it may be shown that

$$\bar{u}_s = \bar{q}_s \sin \psi \cos (\sigma + \vartheta - 90) + \bar{q}_s \cos \psi \sin (\sigma + \vartheta - 90) \tag{B16}$$

$$\bar{v}_s = \bar{q}_s \cos \psi \cos (\sigma + \vartheta - 90) - \bar{q}_s \sin \psi \sin (\sigma + \vartheta - 90) \tag{B17}$$

Equations B5 and B6 are again used to eliminate ψ to obtain the final forms:

$$\bar{u}_s = \bar{U}_\infty \cos\sigma \sin(\sigma + \vartheta) - \epsilon\, \bar{U}_\infty \sin\sigma \cos(\sigma + \vartheta) \tag{B18}$$

$$\bar{v}_s = \epsilon \bar{U}_\infty \sin\sigma \sin(\sigma + \vartheta) + \bar{U}_\infty \cos\sigma \cos(\sigma + \vartheta) \tag{B19}$$

The angular derivatives of the shock variables are required in some of the equations of motion. Notice that the shock variables are functions of the local shock angle σ and derivatives with respect to are written:

$$\frac{d\bar{p}_s}{d\vartheta} = \frac{d\bar{p}_s}{d\sigma} \; \frac{d\sigma}{d\vartheta} \tag{B20}$$

and
$$\frac{d^2\bar{p}_s}{d\vartheta^2} = \frac{d^2\bar{p}_s}{d\sigma^2} \left(\frac{d\sigma}{d\vartheta}\right)^2 + \frac{d\bar{p}_s}{d\sigma} \frac{d^2\sigma}{d\vartheta^2} \tag{B21}$$

where $\bar{p}_s$ has been used as an example. The appropriate derivatives are:

$$\frac{d\bar{u}_s}{d\vartheta} = \bar{v}_s\left(1 + \frac{d\sigma}{d\vartheta}\right) - \bar{U}_\infty \frac{d\sigma}{d\vartheta} \sin\sigma \sin(\sigma + \vartheta) - \bar{U}_\infty \frac{d\epsilon}{d\vartheta} \sin\sigma \cos(\sigma + \vartheta) - \epsilon \frac{d\sigma}{d\vartheta}$$
$$\bar{U}_\infty \cos\sigma \cos(\sigma + \vartheta) \tag{B22}$$

$$\frac{d\bar{v}_s}{d\vartheta} = -\bar{u}_s\left(1 + \frac{d\sigma}{d\vartheta}\right) + \bar{U}_\infty \frac{d\epsilon}{d\vartheta} \sin\sigma \sin(\sigma + \vartheta) + \bar{U}_\infty\, \epsilon \frac{d\sigma}{d\vartheta} \cos\sigma \sin(\sigma + \vartheta)$$
$$-\bar{U}_\infty \frac{d\sigma}{d\vartheta} \sin\sigma \cos(\sigma + \vartheta) \tag{B23}$$

$$\frac{d\bar{\rho}_s}{d\vartheta} = - \frac{\bar{\rho}_\infty}{\epsilon^2} \frac{d\epsilon}{d\vartheta}$$
$$\frac{d\epsilon}{d\vartheta} = \frac{\Gamma \bar{p}_\infty \cos\sigma \left(\frac{8}{7}\epsilon - 1\right)}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^3\sigma \left[\frac{4}{7}\left(\frac{\Gamma \bar{p}_\infty}{\bar{\rho}_\infty \bar{U}_\infty^2 \sin^2\sigma} + 1\right) - \epsilon\right]} \; \frac{d\sigma}{d\vartheta} \tag{B24}$$

$$\frac{d\bar{p}_s}{d\vartheta} = (\bar{p}_s - \bar{p}_\infty) \frac{d\sigma}{d\vartheta} \left[2 \cot\sigma - \frac{1}{1-\epsilon} \frac{d\epsilon}{d\sigma}\right] \tag{B25}$$

$$\frac{d^2\bar{u}_s}{d\vartheta^2} = \frac{d\bar{v}_s}{d\vartheta} - a \frac{d^2\sigma}{d\vartheta^2} - b\left(\frac{d\sigma}{d\vartheta}\right)^2 + \oplus \frac{d\sigma}{d\vartheta} \tag{B26}$$

$$a = -\bar{v}_s + \bar{U}_\infty \sin\sigma \sin(\sigma+\vartheta) + \epsilon\bar{U}_\infty \cos\sigma\cos(\sigma+\vartheta) + \bar{U}_\infty \frac{d\epsilon}{d\sigma}\sin\sigma\cos(\sigma+\vartheta)$$

$$b = \bar{u}_s + \bar{U}_\infty \frac{d^2\epsilon}{d\sigma^2}\sin\sigma\cos(\sigma+\vartheta) + \bar{U}_\infty \sin\sigma\cos(\sigma+\vartheta) - \epsilon\bar{U}_\infty\cos\sigma\sin(\sigma+\vartheta)$$

$$- \bar{U}_\infty \frac{d\epsilon}{d\sigma}\sin\sigma\sin(\sigma+\vartheta) + 2\frac{d\epsilon}{d\sigma}\bar{U}_\infty \cos\sigma\cos(\sigma+\vartheta)$$

$$\oplus = \frac{d\bar{v}_s}{d\vartheta} + \epsilon\bar{U}_\infty\cos\sigma\sin(\sigma+\vartheta) - \bar{U}_\infty\sin\sigma\cos(\sigma+\vartheta) + \bar{U}_\infty\frac{d\epsilon}{d\sigma}\sin\sigma\sin(\sigma+\vartheta)$$

$$\frac{d^2\bar{v}_s}{d\vartheta^2} = -\frac{d\bar{u}_s}{d\vartheta} + e\frac{d^2\sigma}{d\vartheta^2} + f\left(\frac{d\sigma}{d\vartheta}\right)^2 + g\frac{d\sigma}{d\vartheta} \tag{B27}$$

$$e = -\bar{u}_s + \bar{U}_\infty\left[\frac{d\epsilon}{d\sigma}\sin\sigma\sin(\sigma+\vartheta) + \epsilon\cos\sigma\sin(\sigma+\vartheta) - \sin\sigma\cos(\sigma+\vartheta)\right]$$

$$f = -\bar{v}_s + \bar{U}_\infty\left[\frac{d^2\epsilon}{d\sigma^2}\sin\sigma\sin(\sigma+\vartheta) + \frac{d\epsilon}{d\sigma}\sin\sigma\cos(\sigma+\vartheta) + \epsilon\cos\sigma\cos(\sigma+\vartheta)\right.$$

$$\left. + \sin\sigma\sin(\sigma+\vartheta) + 2\frac{d\epsilon}{d\sigma}\cos\sigma\sin(\sigma+\vartheta)\right]$$

$$g = -\frac{d\bar{u}_s}{d\vartheta} + \bar{U}_\infty\left[\frac{d\epsilon}{d\sigma}\sin\sigma\cos(\sigma+\vartheta) + \epsilon\cos\sigma\cos(\sigma+\vartheta) + \sin\sigma\sin(\sigma+\vartheta)\right]$$

$$\frac{d^2\bar{\rho}_s}{d\vartheta^2} = \left[\frac{2\bar{\rho}_\infty}{\epsilon^3}\left(\frac{d\epsilon}{d\sigma}\right)^2 - \frac{\bar{\rho}_\infty}{\epsilon}\frac{d^2\epsilon}{d\sigma^2}\right]\left[\frac{d\sigma}{d\vartheta}\right]^2 - \frac{\bar{\rho}_\infty}{\epsilon}\frac{d\epsilon}{d\sigma}\frac{d^2\sigma}{d\vartheta^2}$$

$$\frac{d^2\epsilon}{d\sigma^2} = \frac{8}{7}\left(\frac{d\epsilon}{d\sigma}\right)\frac{1}{(8/7\epsilon-1)} + \frac{d\epsilon}{d\sigma}\left[\frac{8/7\ \dfrac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty\bar{U}_\infty^2\sin^3\sigma}\cos\sigma + \dfrac{d\epsilon}{d\sigma}}{4/7\left(\dfrac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty\bar{U}_\infty^2\sin^2\sigma} + 1\right) - \epsilon}\right] \tag{B28}$$

$$-\frac{[1+2\cos^2\sigma]}{\sin^2\sigma}\ \frac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty\bar{U}_\infty^2\sin^2\sigma}\ \frac{\left[\frac{8}{7}\epsilon - 1\right]}{\left[\frac{4}{7}\left(\dfrac{\Gamma\bar{p}_\infty}{\bar{\rho}_\infty\bar{U}_\infty^2\sin^2\sigma} + 1\right) - \epsilon\right]}$$

$$\frac{d^2\bar{p}_s}{d\vartheta^2} = \boxplus\frac{d^2\sigma}{d\vartheta^2} + I\left[\frac{d\sigma}{d\vartheta}\right]^2 \tag{B29}$$

$$\boxplus = (\bar{p}_s - \bar{p}_\infty)\left[2 - \frac{1}{1-\epsilon}\frac{d\epsilon}{d\sigma}\right]$$

$$I = (\bar{p}_s - \bar{p}_\infty)\left[2(\cot^2\sigma - 1) - \frac{1}{1-\epsilon}\left(\frac{d^2\epsilon}{d\sigma^2} + 4\cot\sigma\frac{d\epsilon}{d\sigma}\right)\right]$$

APPENDIX C

Continuity and Momentum Equations in Final Form

The final forms of the continuity and $\bar{y}$-momentum equations are presented here. None of the intermediate algebra is given as it only lengthens the appendix. All notation is as introduced previously and all new notation is defined as it is introduced.

The final form of the continuity equation provides a solution for $\bar{u}_b$ and therefore, the continuity equation is effectively solved for it's derivative. Then:

$$\bar{\rho}_b\bar{\delta}\sin\vartheta\left[1-\frac{\bar{u}_b^2}{\Gamma\frac{d\bar{p}_b}{d\bar{\rho}_b}}\right]\frac{d\bar{u}_b}{d\vartheta}=-\bar{\delta}\bar{\rho}_b\bar{u}_b\cos\vartheta-\bar{\delta}\bar{\rho}_s\bar{u}_s\left(\frac{4+3\bar{\delta}}{2}\right)\cos\vartheta-3(1+\bar{\delta})^2$$

$$\bar{\rho}_s\bar{v}_s\sin\vartheta-\delta\left(\frac{4+3\bar{\delta}}{2}\right)\sin\vartheta\left[\bar{\rho}_s\frac{d\bar{u}_s}{d\vartheta}+\bar{u}_s\frac{d\bar{\rho}_s}{d\vartheta}\right]+\bar{\delta}\bar{\rho}_s\bar{u}_s\sin\vartheta-\sin\vartheta\frac{d\bar{\delta}}{d\vartheta}$$

$$[\bar{\rho}_b\bar{u}_b-(1+\bar{\delta})\bar{\rho}_s\bar{u}_s]+\frac{\bar{\delta}^2}{2}(1+\bar{\delta})\sin\vartheta\frac{d}{d\vartheta}\left[\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{y}}\right]_s+\frac{\bar{\delta}}{2}\left[\frac{\partial\bar{\rho}\bar{u}}{\partial\bar{y}}\right]_s[\bar{\delta}(1+\bar{\delta})\cos\vartheta+$$

$$\bar{\delta}\sin\vartheta\frac{d\bar{\delta}}{d\vartheta}+2(1+\bar{\delta})\sin\vartheta] \qquad \text{(C1)}$$

The y-momentum equation may be solved for the second derivative of σ. This is a very complicated equation and no detail of its derivation is given here.

$$\frac{d^2\sigma}{d\vartheta^2}=\frac{1}{R}\left[R_1-R_2\left(\frac{d\sigma}{d\vartheta}\right)^2+R_3\frac{d\sigma}{d\vartheta}+\frac{6}{\delta^2(1+\delta)}(R_4-R_5+R_6)+\Omega R_7\right.$$

$$\left.-WR_8+zR_9-R_{10}\right] \qquad \text{(C2)}$$

where:

$R = a\Omega\bar{\rho}_s\bar{u}_s - \Gamma\Omega$ [illegible] $-We\,\bar{\rho}_s\bar{u}_s + za\bar{\rho}_s + z\frac{\bar{u}_s}{\epsilon}\bar{\rho}_\infty\frac{d\epsilon}{d\sigma}$

$$\Omega=\frac{\bar{v}_sA}{cF}+\frac{(1+\bar{\delta})\bar{u}_s}{cF}\left[4\Gamma\bar{p}_s\frac{d\bar{\delta}}{d\vartheta}-\bar{u}_s\bar{c}\bar{\rho}_s\right]$$

$$z = 4\Gamma\, \bar{p}_s \bar{c}\; (1+\bar{\delta})\; \frac{\bar{u}_s}{cF} - \frac{\bar{v}_s B}{F}$$

$$W = \frac{\bar{v}_s D}{cF} + \frac{\bar{u}_s}{cF}\, [4\Gamma \bar{p}_s\; (\frac{d\delta}{d\vartheta})^2 - \bar{\rho}_s \bar{u}_s \bar{c}\; \frac{d\bar{\delta}}{d\vartheta} - 4\; \rho_s \bar{c}^2]$$

$$R_1 = \frac{2\cot\vartheta}{\bar{\delta}\;(1+\bar{\delta})}\; [\frac{4+3\bar{\delta}}{2}\;\; \bar{\rho}_s \bar{u}_s \bar{v}_s - \bar{\delta}\; \frac{(1+\bar{\delta})}{2}\;\; (\frac{\partial \bar{\rho}\bar{u}\bar{v}}{\partial \bar{y}})_s]$$

$$R_2 = b\Omega\; (\; \bar{\rho}_s \bar{u}_s) - \Gamma I\, \Omega\;\; - Wf\bar{\rho}_s \bar{u}_s - bz\bar{\rho}_s - z\bar{u}_s\; [\frac{2\bar{\rho}_\infty}{\epsilon^3}\; (\frac{d\epsilon}{d\sigma})^2 - \frac{\bar{\rho}_\infty}{\epsilon^2}\, \frac{d^2\epsilon}{d\sigma^2}]$$

$$R_3 = \oplus\, \Omega\, \bar{\rho}_s \bar{u}_s + gW\bar{\rho}_s \bar{u}_s - \oplus\, \bar{\rho}_s z$$

$$R_4 = \frac{1}{3}\, \frac{d\bar{\delta}}{d\vartheta} [\frac{4+3\bar{\delta}}{2}\; \bar{\rho}_s \bar{u}_s \bar{v}_s - \frac{\bar{\delta}(1+\bar{\delta})}{2}\; (\frac{\partial \bar{\rho}\bar{u}\bar{v}}{\partial \bar{y}})_s] + \bar{\delta}\bar{\rho}_s \bar{u}_s \bar{v}_s\, \frac{d\bar{\delta}}{d\vartheta} + \frac{\bar{\delta}}{3}\, \frac{(4+3\delta)}{2}\;\; [\bar{\rho}_s \bar{u}_s\, \frac{d\bar{v}_s}{d\vartheta}$$

$$+ \bar{\rho}_s \bar{v}_s\, \frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s \bar{v}_s\, \frac{d\bar{\rho}_s}{d\vartheta}]$$

$$R_5 = \frac{\bar{\delta}}{6}\; (1+2\bar{\delta})\; \frac{d\bar{\delta}}{d\vartheta} [\frac{\partial \bar{\rho}\bar{u}\bar{v}}{\partial \bar{y}}]_s - \frac{\bar{\delta}}{3}\; [\frac{4+3\bar{\delta}}{2}\; \bar{\rho}_s \bar{u}_s^2 + \bar{\rho}_b \bar{u}_b^2 - \frac{\bar{\delta}(1+\bar{\delta})}{2}\; (\frac{\partial \rho u^2}{\partial \bar{y}})_s]$$

$$R_6 = (1+\bar{\delta})^2 [\Gamma \bar{p}_s + \bar{\rho}_s \bar{v}_s^2] - (1+\bar{\delta})\bar{\rho}_s \bar{u}_s \bar{v}_s\, \frac{d\bar{\delta}}{d\vartheta} - \Gamma \bar{p}_b - \frac{\Gamma\bar{\delta}}{3} [(4+\bar{\delta})\bar{p}_b - 2(1+\bar{\delta})\bar{p}_s + \bar{\delta}(1+\bar{\delta})$$

$$\frac{\bar{\rho}_b \bar{u}_b^2}{\Gamma}\;]$$

$$R_7 = 2\bar{\rho}_s \bar{u}_s\, \frac{d\bar{v}_s}{d\vartheta} + \frac{d\bar{u}_s}{d\vartheta}\;\; (\bar{\rho}_s \bar{v}_s + \bar{\rho}_s\, \frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\, \frac{d\bar{\rho}_s}{d\vartheta}) + \bar{u}_s \bar{v}_s\, \frac{d\bar{\rho}_s}{d\vartheta}$$

$$R_8 = 2\bar{\rho}_s \bar{u}_s\, \frac{d\bar{u}_s}{d\vartheta} + [\bar{\rho}_s\, \frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\, \frac{d\bar{\rho}_s}{d\vartheta}][\bar{u}_s - \frac{d\bar{v}_s}{d\vartheta}]$$

$$R_9 = 3\bar{\rho}_s\, \frac{d\bar{v}_s}{d\vartheta} + 2\, \frac{d\bar{\rho}_s}{d\vartheta}\, (\frac{d\bar{u}_s}{d\vartheta} + \bar{v}_s) + \cot\vartheta\;\; (\bar{\rho}_s\, \frac{d\bar{u}_s}{d\vartheta} + \bar{u}_s\, \frac{d\bar{\rho}_s}{d\vartheta}) - \frac{\bar{\rho}_s \bar{u}_s}{\sin^2\vartheta}]$$

$$R_{10} = \frac{\bar{v}_s}{cF}\; [\; \square_s \frac{dA}{d\vartheta} - \Delta_s \frac{d(Bc)}{d\vartheta} + \nabla_s \frac{dD}{d\vartheta} - cE\, \frac{d\Phi_s}{d\vartheta} - \Phi_s\, \frac{d(CE)}{d\vartheta} - (c\; \frac{dF}{d\vartheta} + F\; \frac{dc}{d\vartheta})$$

$$(\frac{\partial \rho u}{\partial \bar{y}})_s] + [\frac{\partial \rho u}{\partial \bar{y}}]_s\, \frac{d\bar{v}_s}{d\vartheta} + [\frac{1}{\bar{u}_s}\; \frac{d\bar{u}_s}{d\vartheta} - \frac{1}{F}\, \frac{dF}{d\vartheta} - \frac{1}{c}\, \frac{dc}{d\vartheta}][(\frac{\partial \bar{\rho}\bar{u}\bar{v}}{\partial \bar{y}})_s - \bar{v}_s (\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{y}})_s]$$

$$+ \frac{\bar{u}_s \Box_s}{cF}\Big\{4\Gamma\,[(1+\bar{\delta})\,(\bar{p}_s \frac{d^2\bar{\delta}}{d\vartheta^2} + \frac{dp_s}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta}) + \bar{p}_s\,(\frac{d\bar{\delta}}{d\vartheta})^2] - (1+\bar{\delta})\,[\bar{u}_s c \frac{d\bar{\rho}_s}{d\vartheta} + \bar{u}_s \bar{\rho}_s \frac{dc}{d\vartheta}$$

$$+ \bar{\rho}_s\, c\, \frac{d\bar{u}_s}{d\vartheta}] - \bar{u}_s\, c\, \bar{\rho}_s \frac{d\bar{\delta}}{d\vartheta}\Big\} \quad + \frac{4\Gamma \bar{u}_s \Delta_s}{cF}[\bar{p}_s c \frac{d\bar{\delta}}{d\vartheta} + c\,(1+\bar{\delta})\,\frac{d\bar{p}_s}{d\vartheta} + \bar{p}_s\,(1+\bar{\delta})\,\frac{dc}{d\vartheta}]$$

$$+ \frac{\bar{u}_s}{F}\,[2(1+\bar{\delta})\,\Phi_s \frac{dc}{d\vartheta} + c\,(\Phi_s \frac{d\bar{\delta}}{d\vartheta} + (1+\bar{\delta})\,\frac{d\Phi_s}{d\vartheta})] + \frac{\bar{u}_s \nabla_s}{cF}\,[4\Gamma(2\bar{p}_s \frac{d^2\bar{\delta}}{d\vartheta^2}\frac{d\bar{\delta}}{d\vartheta}$$

$$+ \frac{d\bar{p}_s}{d\vartheta}\,(\frac{d\bar{\delta}}{d\vartheta})^2) - \bar{\rho}_s \bar{u}_s c \frac{d^2\bar{\delta}}{d\vartheta^2} - \bar{\rho}_s \bar{u}_s \frac{dc}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta} - \bar{\rho}_s c \frac{d\bar{\delta}}{d\vartheta}\frac{d\bar{u}_s}{d\vartheta} - \bar{u}_s c \frac{d\bar{\rho}_s}{d\vartheta}\frac{d\bar{\delta}}{d\vartheta}$$

$$- 8\bar{\rho}_s c \frac{dc}{d\vartheta} - 4c^2 \frac{d\bar{\rho}_s}{d\vartheta}]$$